Extract from

On the principles of translating algebraic quantities into probable relations and annuities, &c.

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Cambridge, 1792

LEMMA.

1. Let $\frac{A+Br+Cr^2+Dr^3+\cdots+Hr^{l-1}+\cdots+r^{z-2}}{(1-r)^z} = 1+2^{z-1}\times r+3^{z-1}\times r^2+4^{z-1}\times r^3+5^{z-1}\times r^4+6^{z-1}\times r^5+$ &c. in infinitum; then will

$$\begin{aligned}
A &= 1, \\
B &= 2^{z-1}-z, \\
C &= 3^{z-1}-z\times 2^{z-1}+z\cdot\frac{z-1}{2}, \\
D &= 4^{z-1}-z\times 3^{z-1}+z\cdot\frac{z-1}{2}\times 2^{z-1}-z\cdot\frac{z-1}{2}\cdot\frac{z-2}{3}, \\
&\vdots \\
H &= l^{z-1}-z\times(l-1)^{z-1}+z\cdot\frac{z-1}{2}\times(l-2)^{z-1}-z\cdot\frac{z-1}{2}\cdot\frac{z-2}{3}\times(l-3)^{z-1} \\
&\quad +z\cdot\frac{z-1}{2}\cdot\frac{z-2}{3}\cdot\frac{z-3}{4}\times(l-4)^{z-1}-\text{\&c.}
\end{aligned}$$

The coefficient of the last term r^{z-2} will be 1, and all the subsequent coefficients become $=0$, as is manifest from a theorem published in the *Phil. Trans.* viz. $l^{z-1}-z(l-1)^{z-1}+z\cdot\frac{z-1}{2}(l-2)^{z-1}-$ &c. will be $=1$, when $l=z-1$; but $=0$, when l is greater than $z-1$.

The same principles may be applied when z is either negative or fractional, in which case the series $A+Br+Cr^2+Dr^3+$ &c. proceeds in infinitum.

If for r in the above mentioned quantities be substituted $-r$, then the terms resulting will be alternately $+$ and $-$.

From the lemma and the principles given in the *Meditat. Algebr.* which were sent in some papers to the Royal Society in 1757, may be found the sum of every (s) terms of the series $1+2^{z-1}\times r+3^{z-1}\times r^2+$ &c.

2. *To find the sum of the series* $1+2^{z-1}r+3^{z-1}\times r^2+4^{z-1}\times r^3+\cdots+(k-1)^{z-1}\times r^{k-2}$ *consisting of* $k-1$ *terms.*

The sum will be equal to a fraction, of which the denominator is (as in the preceding lemma) $(1-r)^z$, and the numerator

$$\begin{array}{rl}
A & + \quad Br+Cr^2+Dr^3+\cdots+Hr^{l-1}+\cdots+r^{z-2}-k^{z-1}\times r^{k-1} \\
& - \quad \left((k+1)^{z-1}-z\times k^{z-1}\right)\times r^k \\
& - \quad \left((k+2)^{z-1}-z\times(k+1)^{z-1}+z\cdot\dfrac{z-1}{2}\times k^{z-1}\right)\times r^{k+1} \\
& - \quad \left((k+3)^{z-1}-z\times(k+2)^{z-1}+z\cdot\dfrac{z-1}{2}\times(k+1)^{z-1}\right. \\
& \qquad \left.-z\cdot\dfrac{z-1}{2}\cdot\dfrac{z-2}{3}\times k^{z-1}\right)\times r^{k+2} \\
& - \quad \left((k+4)^{z-1}-z\times(k+3)^{z-1}+z\cdot\dfrac{z-1}{2}\times(k+2)^{z-1}\right. \\
& \qquad \left.-z\cdot\dfrac{z-1}{2}\cdot\dfrac{z-2}{3}\times(k+1)^{z-1}+z\cdot\dfrac{z-1}{2}\cdot\dfrac{z-2}{3}\cdot\dfrac{z-3}{4}\times k^{z-1}\right)\times r^{k+3} \\
& - \quad \&c.
\end{array}$$

to z terms; all the remaining terms, by the *Phil. Trans.*, become $=0$: the coefficients A, B, C, D,… H to $z-1$ terms denote the same quantities as in the preceding lemma; and the law of the z terms may easily be seen from inspection, or collected from the law of the terms in the preceding lemma.

3. This lemma may easily be deduced from the preceding by the following proposition.

Let d_1, $d_2\times r$, $d_3\times r^2$, $d_4\times r^3$, &c. be the successive terms of the divisor of any fraction; and q_1, $q_2\times r$, $q_3\times r^2$, $q_4\times r^3$, &c. the successive terms of the quotient, and $R.h\times r^h$, $R\overline{h+1}\times r^{h+1}$, $R\overline{h+2}\times r^{h+2}$, $R\overline{h+3}\times r^{h+3}$, &c. the successive terms of the remainder, of which the distance from the first is $h-1$, that is, let the divisor be $d_1+d_2\times r+d_3\times r^2+d_4\times r^3+$&c., the quotient $q_1+q_2\times r+q_3\times r^2+q_4\times r^3+$&c., and the remainder (whose distance from the first is $h-1$) $R.h\times r^h+R\overline{h+1}\times r^{h+1}+R\overline{h+2}\times r^{h+2}R\overline{h+3}\times r^{h+3}+$&c.; then will

$$\begin{array}{rcl}
Rh & = & d_1\times q.\overline{h+1}, \\
R.\overline{h+1} & = & d_1\times q.\overline{h+2}+d_2\times q.\overline{h+1}, \\
R.\overline{h+2} & = & d_1\times q.\overline{h+3}+d_2\times q.\overline{h+2}+d_3\times q.\overline{h+1}, \\
R.\overline{h+3} & = & d_1\times q.\overline{h+4}+d_2\times q.\overline{h+3}+d_3\times q.\overline{h+2}+d_4\times q.\overline{h+1}, \\
R.\overline{h+4} & = & d_1\times q.\overline{h+5}+d_2\times q.\overline{h+4}+d_3\times q.\overline{h+3}+d_4\times q.\overline{h+2}+d_5\times q.\overline{h+1}, \\
\&c. & &
\end{array}$$

Hence the terms of the remainders may easily be deduced from the terms of the divisor and quotient.

Cor. If the terms of the quotient contained between two different remainders be required, subtract the last remainder from the first, and the difference divided by the divisor will give the terms of the quotient required.

Cor. Hence the sum of any series

$$\begin{aligned}&(a+b+c+d+\&\text{c.})\\&\quad+(a\times 2^{z-1}+b\times 2^{z-2}+c\times 2^{z-3}+d\times 2^{z-4}+\&\text{c.})r\\&\quad+(a\times 3^{z-1}+b\times 3^{z-2}+c\times 3^{z-3}+\&\text{c.})r^2\\&\quad+(a\times 4^{z-1}+b\times 4^{z-2}+c\times 4^{z-3}+\&\text{c.})r^3\\&\quad+\&\text{c.}\end{aligned}$$

may be deduced; for the sum of each of the terms multiplied into a is found by the lemma; and in like manner may be found the sum of each of the terms multiplied into b, into c, &c.; and consequently by adding them together the sum of the whole series.

Of translating propositions expressing the relation between algebraical quantities to others expressing probable relations.

1. Certainly being contained in the whole number of events possible, may be expressed by any given quantity or number; e.g., N or 1.

2. The probability or chance $\left(\frac{a}{N}\text{ or }a\right)$ will always be less than the quantity or number abovementioned N or 1; and consequently its reciprocal greater than $\frac{1}{N}$ or 1.

3. Let a and b be the chances of two events A and B independent of each other respectively succeeding, and certainty N, then will $\frac{a\times b}{N^2}$ be the chance of both the events succeeding in the two trials; and $\frac{\overline{N-b}\times a}{N^2}$ be the chance of B's failing in one and A's succeeding in the other, and $\frac{b\times\overline{N-a}}{N^2}$ the chances of B's succeeding and A's failing, and $\frac{\overline{N-a}.\overline{N-b}}{N^2}$ the chance of A's and B's both failing.

For, if $\frac{a}{N}$ be the chance of A's succeeding, $\frac{N-a}{N}$ will be the chance of its failing in the same trial.

4. Let a, b, c, d, &c.; α, β, γ, δ, &c. be the chances of any independent events A, B, C, D, &c.; A'. B', Γ, Δ, &c. succeeding to certainty N; then will the chance of all the events succeeding be

$$\frac{a}{N}\times\frac{b}{N}\times\frac{c}{N}\times\frac{d}{N}\times\&\text{c.}\times\frac{\alpha}{N}\times\frac{\beta}{N}\times\frac{\gamma}{N}\times\frac{\delta}{N}\times\&\text{c.}$$

The chance of the events A, B, C, D, &c. succeeding, and the events A', B', Γ, Δ, &c. failing will be

$$\frac{a}{N}\times\frac{b}{N}\times\frac{c}{N}\times\frac{d}{N}\times\&\text{c.}\times\frac{N-\alpha}{N}\times\frac{N-\beta}{N}\times\frac{N-\gamma}{N}\times\&\text{c..}$$

5. The chance of the event of A succeeding n times together, and of B succeeding m times, and C happening r times, &c,, and of A' failing n' times together, and B' failing m' times, and Γ failing r' times, &c. will be

$$\frac{a^n}{N^n}\times\frac{b^m}{N^m}\times\frac{c^r}{N^r}\times\&\text{c.}\times\left(\frac{N-\alpha}{N}\right)^{n'}\times\left(\frac{N-\beta}{N}\right)^{m'}\times\left(\frac{N-\gamma}{N}\right)^{r'}\times\&\text{c.}$$

These events are supposed to be independent of each other, or to happen in a given order.

6. If the order of two dependent events, as A and B, be not fixed; i.e., the event A may happen first and B second, or B may happen first and A second; then will the chance of the events A and B happening in the two trials be $2 \times \frac{a}{N} \times \frac{b}{N}$.

In the same manner the chance B's happening m times, C's happening r times, and A's failing n' times, B' failing m' times, Γ failing r' times, &c. in $r + n' + m' + r' +$ &c. $= l$ trials in every order possible, will be

$$l \times \frac{l-1}{2} \times \frac{l-2}{3} \cdots \frac{l-m+1}{m} \times \frac{l-m}{1} \times \frac{l-m-1}{2} \times \frac{l-m-2}{3} \cdots$$
$$\frac{l-m-r+1}{r} \times \frac{l-m-r}{1} \times \frac{l-m-r-1}{2} \times \frac{l-m-r-2}{3} \cdots$$
$$\frac{l-m-r-n'+1}{n'} \times \frac{l-m-r-n'}{1} \times \frac{l-m-r-n'-1}{2} \times \frac{l-m-r-n'-2}{3} \cdots$$
$$\frac{l-m-r-n'-m'+1}{m'} \times \frac{l-m-r-n'-m'}{1} \times \frac{l-m-r-n'-m'-1}{2}$$
$$\times \frac{l-m-r-n'-m'-2}{3} \cdots \frac{l-m-r-n'-m'-r'+1}{r'} \times \text{\&c.}$$
$$(H) \times \frac{b^m}{N^m} \times \frac{c^r}{N^r} \times \left(\frac{N-\alpha}{N}\right)^{n'} \times \left(\frac{N-\beta}{N}\right)^{m'} \times \left(\frac{N-\gamma}{N}\right)^{r'} \times \text{\&c.}$$

7. From these principles any quantities may be translated into the probabilities of events. e.g. 1. any power $\frac{a^n}{N^n} \times H$ may be translated into the probability of an event's happening n times (a whole number) of which the chance of its happening each time is $\frac{a}{N}$, multiplied into H; in this case a is supposed less than N: but if a be greater than N, it must be construed the reciprocal of the abovementioned probability. 2.$\left(\frac{a}{N}\right)^{\frac{1}{m}}$ may be construed the probability or chance of an event's succeeding in one time, of which the chance of its happening (m) times together is $\frac{a}{N}$. 3. $\left(\frac{a}{N}\right)^{\frac{n}{m}}$ denotes the chance of an event's happening (n) times together, of which the chance of its happening (m) times together is $\frac{a}{N}$.

4. $H \times \frac{a^n}{N^n} \times \frac{b^m}{N'^m}$ may be construed the chance of the event's (A) happening n times together, and the event's (B) happening (m) times together (of which the respective chances of happening each time are $\frac{a}{N}$ and $\frac{b}{N'}$) multiplied into H, if the events are supposed independent of each other, or to happen in a given order.

5. The quantity $\left(H + Ka^{\frac{n}{m}} \times b^p + L \times a^{\frac{b}{k}}\right)^{\frac{r}{s}}$ may denote the chance of an event (S) happening (r) times together, of which the chance of its happening (s) times together is $H + Ka^{\frac{n}{m}} \times b^p + L \times a^{\frac{b}{k}}$; which quantity may be translated into probable relations from the principles before given.

In the same manner may all algebraical and exponential quantities, and their nascent or finite increments be translated into similar quantities. If a be greater than N, or b than N'; then for the probabilities or chances of happening or failing always read their reciprocals.

8. Any algebraical, &c. equations may be translated from these principles into equations expressing the relation between quantities of this kind, and vice versa, any

equations expressing the relations between quantities of this kind may in like manner be translated into algebraical, &c. equations.

9. Hence any algebraical equations containing (l) unknown quantities, may be translated into equations expressing the relation between $(l-r)$ unknown quantities and (r) different independent chances; and these last mentioned equations may be translated into the former.

10. Any given equations expressing the relations between algebraical, &c. quantities may, by the preceding methods, be translated into innumerable different equations; for every unknown quantity, its powers, &c. contained in the given equations, may be substituted the chances or probabilities of an event's succeeding for a certain number of times, or its not succeeding; or a compound function of both; and the resolutions, reductions, &c. of the resulting equations can easily be deduced from the resolutions, reductions, &c. of the given ones.

11. e.g. Let the chance of an event's succeeding, contained in the given equations, be $\frac{a}{N}$, then if in all the terms of the given equations for a be substituted $N-a'$, there will result equations, in which a is exterminated, and a' the chance of its not succeeding is substituted in its place: but if in some of the terms of the equations be substituted $N-a'$ for a, and in others not, there will result different equations, in which are contained both a and a', &c.

12. If the same algebraical quantities, &c. are exprest in different algebraical, &c. formulæ, then by translating them into probable relations, as is before taught, the equality will be translated into a proposition concerning chances.

Ex. 1. $\frac{a}{N}\times\frac{N-a}{N}=\frac{a}{N}-\frac{a^2}{N^2}$; let $\frac{a}{N}$ be the chance of an event's succeeding in one trial, then will $\frac{N-a}{N}$ be the chance of its failing, and $\frac{a^2}{N^2}$ the chance of its happening twice together; and therefore the chance of its happening in the first trial and failing in the second will be equal to the difference between the chance of the event's succeeding in the first trial and its succeeding twice together: but if the order of its happening and failing is not given, then the one half of the chance of its happening once and failing once in two trials will be equal to the abovementioned difference.

Ex. 2. $\frac{a+b}{N}\times\frac{a-b}{N}=\frac{a^2-b^2}{N^2}$, and consequently the chance of an event's happening, of which the probability is $\frac{a+b}{N}$, and then an event happening, of which the probability is $\frac{a-b}{N}$, is equal to the difference between the chances of two different events respectively happening twice in two trials, of which then respective chances for happening in one trial are $\frac{a}{N}$ and $\frac{b}{N}$.

Ex. 3. $\frac{\overline{a+b}^n}{N^n}=\frac{a^n+a^{n-1}b+n\cdot\frac{n-1}{2}a^{n-2}b^2\cdots+n\cdot\frac{n-1}{2}a^2b^{n-2}+nab^{n-1}+b^n}{N^n}$, and consequently the chance of an event's happening n times in n trials, of which the chance of its happening in one trial is $\frac{a+b}{N}$, will be equal to the sum of the chances of two events A and B happening n times in n trials, and of the chances of the one happening $n-1$ times and the other one time in the abovementioned trials, and of the chances of the one happening $n-2$ times and the other two times in the same trials, &c. when the order of the event's happening in is not given; the chances of the events A and B happening in one trial being respectively $\frac{a}{N}$ and $\frac{b}{N}$: but if the order of the abovementioned events be given, then will the abovementioned chance of an event's (of which the chances of happening in one trial is $\frac{a+b}{N}$) happening n times in n trials will be equal to the sum of the chances of the events A and B happening n times in n trials $+n\times$ sum of the chances of the one happening $n-1$ times and the other one time $+n\times\frac{n-1}{2}\times$ sum of the one happening $n-2$ and the other two times, &c. in the same number (n) of trials.

Ex. 4. $\frac{N}{N-a}=1+\frac{a}{N}+\frac{a^2}{N^2}+\frac{a^3}{N^3}+$ &c. in infinitum; but $\frac{N-a}{N}$ is the probability of an event's failing, of which the probability of an event's failing, of which the probability of its happening is $\frac{a}{N}$; therefore the reciprocal of the probability of an event's failing, of which the probability of its happening is $\frac{a}{N}$, will exceed 1 by the sum of the probabilities of the abovementioned events happening once in one trial, twice in two trials, three times in three trials, four times in four, &c. in infinitum.

Ex. $\frac{e}{a+b}=\frac{e}{a}-\frac{e}{a}\left(\frac{b}{a}-\frac{b^2}{a^2}+\frac{b^3}{a^3}-\frac{b^4}{a^4}+\right.$&c.; suppose three events A, B, C, of which the respective chances for happening in each trial are $\frac{e}{a+b}$, $\frac{e}{a}$ and $\frac{b}{a}$; then the chance of A's happening will be less than the chance of B's happening by the chance of B's happening once multiplied into the difference between the chances of C's happening 1, 3, 5, 7, &c. times and its happening 2, 4, 6, &c. times in infinitum.

Quantities of this kind may be interpolated by interpolating the quantities contained in them, their indexes, &c.; they may be multiplied and divided by each other, proper attention being had to the mode of translation before given.

If two or more different chances may justly be translated into the same algebraical quantity, they may be concluded equal.

A chance may consist of the sum or difference of several chances, which may depend on the happening of several events, and be translated accordingly.

It may not be unworthy of observation, that the translation of quantities into their different equalities or meanings, and decomposing them by different methods into the different parts of which they consist, constitute a considerable share of many sciences. In arithmetic the bare expansion of terms often develops many propositions before concealed: e.g. Let the chances of two events (A and B) happening be to each other $::a:b$; and the sum of all the chances possible, i.e. certainty N; then will the chance of the events A and B happening n times in n trials be

$$\frac{\overline{a+b}^n}{N^n}=\frac{\begin{array}{r}a^n+a^{n-1}b+n\cdot\dfrac{n-1}{2}a^{n-2}b^2\cdots+La^{n-l}b^l+\cdots+Ma^{n-m}b^n+\cdots\\+n\dfrac{n-1}{2}\times a^2b^{n-2}+nab^{n-1}+b^n\end{array}}{N^n}$$

of which the chance of happening in one trial is $\frac{a+b}{N}$.

2. The chance of the events (A)'s happening $(n-l)$ times and (B)'s happening l times will be the term $L\times\frac{a^{n-l}\times b^l}{N^n}$; and the chance of (A)'s happening $(n-l)$ times or more in n trials will be the sum of all the terms to $S\times a^{n-l}\times b^l$ included, divided as before by N^n; and the chance of (A)'s happening fewer times than $(n-l)$ and B's happening more times than (l) will $=$ the remaining terms divided by the same divisor.

3. The chance of the event (A) happening an even number to its happening an odd number of times, will be as the sum of all the alternate terms to each other, in a similar manner may be deduced the chance of its happening every h^{th} time to the chance of its happening every k^{th} time.

4. The chance of (A)'s gaining $(n-l)$ games to B's gaining $(l-1)$ games will be as

$$a^n + na^{n-1}b + n\cdot\frac{n-1}{2}a^{n-2}b^2 + \cdots + La^{r-l}bl$$

to the sum of all the remaining terms, and from the same principles may be deduced the chance of the events (A)'s happening between h and k times to A's or B's happening between l and m times.

5. The chance of A's happening once, twice, thrice, or $\frac{h}{k}$ times for B's happening once in n trials will $= L \times \frac{a^{n-1}\times b^l}{N^n}$, where $n-l = l$, $2l$, $3l$, or $\frac{h}{k}\times l$; and the chance of its happening the abovementioned number of times or more to its not happening $:: a^n + na^{n-1}b + \cdots La^{n-1}b^l$ to the sum of all the remaining terms.

6. The chance of A's winning r games more than B to the chance of his not doing it $:: a^n + na^{n-1}b + \cdots + La^{n-1}b^l$: sum of the remaining terms, where $l = \frac{n-r}{2}$.

Innumerable other like propositions may easily be deduced.

If any of the preceding quantities are required to be equal, or in a given ratio to each other; suppose them equal, &c. and from the resulting equation or equations, by the principles of algebra, approximations, &c. investigate the unknown quantities.

The number of unknown quantities, as is well known, must not exceed the number of equations.

In these problems, the unknown quantities mentioned are a, b, n and N, and if three of them are given, the fourth can be found; if n be the quantity required, the equation will become an exponential one, if the proportion of $a : b$, and not the quantities themselves, be required, for a substitute 1.

2. Let more quantities a, b, c, d, &c. be respective chances of any different events A, B, C, D, &c. to certainty N, and $\frac{a+b+c+d+\text{\&c.}}{N}$ the chance of an event's happening; then will the chance of its happening n times in n trials be

$$\left(\frac{a+b+c+d+\text{\&c.}}{N}\right)^n = \frac{\begin{aligned} a^n + na^{r-1}(b+c+d+\text{\&c.}) + n\cdot\frac{n-1}{2} \\ \times a^{n-2}(b^2+c^2+d^2+\text{\&c.}) \\ +n.\overline{n-1}\times a^{n-2}(bc+bd+cd+\text{\&c.}) \\ +\cdots+La^lb^mc^pd^q\text{\&c.}+\text{\&c.}\end{aligned}}{N^n},$$

and the chance of A's happening l times, B's happening m times, C's happening p times, D's happening q times, &c. will then be n.

$$\begin{aligned}\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-l+1}{l}\times\frac{n-l}{1}\times\frac{n-l-1}{2}\cdot\frac{n-l-2}{3}\cdots \\ \frac{n-l-m+1}{m}\times\frac{n-l-m}{1}\times\frac{n-l-m-1}{2}\cdots\times \\ \frac{n-l-m-p+1}{p}\times\frac{n-l-m-p}{1}\times\frac{n-l-m-p-1}{2}\cdots \\ \frac{n-l-m-p-q+1}{q}\times\text{\&c.}(L)\times a^lb^mc^pd^q\text{\&c.};\end{aligned}$$

where $n = l + m + p + q +$ &c. divided by N^n.

Every corollary contained in the preceding case may be applied to this, and in the terms which include a, b, c, d, &c. equally, as a and b.

Cor. Let

$$\begin{aligned}\left(\frac{a+b+c+d+\text{\&c.}}{N}\right)^n &= \left(\frac{a^1+a^2+a^3+\cdots+a^m}{N}\right)^n \\ &= \frac{a^n+na^{n+1}+\cdots+L\times a^l\ldots+a^{mn}}{N^n},\end{aligned}$$

and consequently if there are (m) events, of which the probabilities of succeeding are equal and $\frac{p}{N}$, marked 1, 2, 3, 4,... m points, then will the chance of the happening of l points in n trials be $\frac{Lp^n}{N^n}$.

3. If some terms favour two or more parties, decompose them into all their simple parts, and add the parts favouring any one to his expectation. e.g. suppose the term $a^m b^{l-1}$; and let A (whose chance for succeeding at every one trial is a) gain, if he succeeds r times before B (whose chance for winning at every trial is b) succeeds s times, where r and s are equal or less than m and $l-1$ respectively, and $C(a^h b^k)$ denote the number of all the different combinations, into which the quantity $a^h b^k$ can be decomposed, viz. $a^h b^k$, $a^{h-1}b^k a$, $a^{h-2}b^{k-1}aba$, $a^{h-2}b^k a^2$, &c. of which the number will be $\overline{h+k}\times\frac{h+k-1}{2}\times\frac{h+k-2}{3}\ldots\frac{h+1}{k}$ or $\frac{k+1}{h}$; then will the number of terms, in which A gains be $C(a^m b^{s-1})\times b^{l-s}+C(a^{m-1}b^{s-1})\times C(ab^{l-s})+C(a^{m-2}b^{s-1})\times C(a^2b^{l-s})+C(a^{m-3}b^{s-1})\times C(a^3b^{l-s})\ldots\times C(a^r b^{s-1})\times C(a^{m-r}b^{l-s})$; since in each of these combinations A gains (r) times before B gains (s) times, but the number of these combinations are

$$\begin{aligned}&\overline{m+s-1}\times\frac{m+s-2}{2}\times\frac{m+s-3}{3}\cdots\frac{s}{m}\left(\text{or }\frac{m+1}{s-1}\right)\\
&\times 1+\overline{m+s-2}\times\frac{m+s-3}{2}\times\frac{m+s-4}{3}\cdots\frac{s}{m-1}\left(\text{or }\frac{m}{s-1}\right)\\
&\times\overline{l-s+1}+\overline{m+s-3}\times\frac{m+s-4}{2}\times\frac{m+s-5}{3}\cdots\frac{s}{m-2}\left(\text{or }\frac{m-1}{s-1}\right)\\
&\times\overline{l-s+2}\times\frac{l-s+1}{2}+\overline{m+s-4}\times\frac{m+s-5}{2}\times\frac{m+s-6}{3}\cdots\frac{s}{m-3}\left(\text{or }\frac{m-2}{s-1}\right)\\
&\times\overline{l-s+3}\times\frac{l-s+2}{2}\times\frac{l-s+1}{3}+\cdots\\
&+\overline{r+s-1}\times\frac{r+s-2}{2}\cdot\frac{r+s-3}{3}\cdots\frac{s}{r}\left(\text{or }\frac{r+1}{s-1}\right)\\
&\times\overline{l-s+1}\times\frac{l-s+2}{2}\cdot\frac{l-s+3}{3}\cdots\frac{l-s+1}{m-r}\left(\text{or }\frac{m-r+1}{l-s}\right).\end{aligned}$$

If in the above quantity for m and r be substituted $l-1$ and s; and vice versa for $l-1$ and s be substituted m and r, the quantity resulting will express the number of terms which favour B.

4.1. If the chance depends on the expectations contained in a number of successive trials: Find the chance of the expectation from the first trials; then find the chance of its failing in the first and succeeding in the second, and so on in infinitum; or the chance of its succeeding in the second to be added to the first, and so on; the sum of the resulting chances will be the chance required.

2. If the several expectations depend on the chances of some events happening and others failing or happening in some given relation to them: Find the chances of each of the events happening and the other failing or happening in the give relation, and so of each of the expectations, and thence the chance required.

Ex. 1. Let the chances of the events A and B happening be respectively $\frac{a}{a+b}$ and $\frac{b}{a+b}$; then the chance of the event A happening r times more than b in r trial will be

$$\frac{a^r}{(a+b)^r},$$

in $r+2$ trials will be

$$\frac{a^r}{(a+b)^r}\left(1+r\times\frac{ab}{(a+b)^2}\right);$$

in $r+4$ trials will be

$$\frac{a^r}{(a+b)^r}\left(1+r\times\frac{ab}{(a+b)^2}+r\cdot\frac{r+3}{2}\times\frac{a^2b2}{(a+b)^4}\right),$$

and in general it will be

$$\begin{aligned}\frac{a^r}{(a+b)^r}\times\Bigg(1+r\times\frac{ab}{(a+b)^2}+r\cdot\frac{r+3}{2}\times\frac{a^2b^2}{(a+b)^4}+r\cdot\frac{r+4}{2}\cdot\frac{r+5}{3}\\ \times\frac{a^3b^3}{(a+b)^6}+r\cdot\frac{r+5}{2}\cdot\frac{r+6}{3}\cdot\frac{r+7}{4}\times\frac{a^4b^4}{(a+b)^8}+\cdots\\ +r\cdot\frac{r+l+1}{2}\cdot\frac{r+l+2}{3}\cdot\frac{r+l+3}{4}\cdots\frac{r+2l-1}{l}\times\frac{a^lb^l}{(a+b)^{2l}}+\cdots+\text{\&c.}\Bigg)\end{aligned}$$

in infinitum.

This may be deduced from the subsequent arithmetical theorem. viz.

$$\begin{aligned}2m\times\frac{2m-1}{2}\cdot\frac{2m-2}{3}\cdots\frac{2m-s}{s+1}+r\times(2m-2)\cdot\frac{2m-3}{2}\cdot\frac{2m-4}{3}\cdots\frac{2m-s-1}{s}\\ +r\cdot\frac{r+3}{2}\times(2m-4)\cdot\frac{2m-5}{2}\cdots\frac{2m-s-2}{s-1}\\ +r\cdot\frac{r+4}{2}\frac{r+5}{3}\times(2m-6)\cdot\frac{2m-7}{2}\cdot\frac{2m-8}{3}\cdots\frac{2m-s-3}{s-2}\\ +\cdots+r\cdot\frac{r+s+2}{2}\cdot\frac{r+s+3}{2}\cdot\frac{r+s+4}{3}\cdots\frac{r+2s+1}{s+1}\\ =(r+2m)\cdot\frac{r+2m-1}{2}\cdot\frac{r+2m-2}{3}\cdot\frac{r+2m-3}{4}\cdots\frac{r+2m-s}{s+1}.\end{aligned}$$

Ex. 2. If it be required to find the chance of A's succeeding n times as oft as B's precisely: in $n+1$ trails it will be found $(n+1)\frac{a^nb}{(a+b)^{n+1})}=P$; in $2n+2$ trials it will be found

$$P+n\cdot(n+1)\times\frac{a^{2n}b^2}{(a+b)^{2n+2}}=Q;$$

in $3n+3$ it will be

$$Q + n \cdot \frac{n+1}{2} \cdot (3n+1) \times \frac{a^{3n}b^3}{(a+b)^{3n+3}} \times \text{\&c.}$$

The same principles may be applied to innumerable examples, and similar propositions may be collected (mutatis mutandis) when applied to the case contained in the following theor.

5.1 Let a, b, c, d, &c. be the respective chances of the happening of α, β, γ, δ, &c.: in one trial (A), and

$$\begin{aligned} & (ax^{\alpha} + bx^{\beta} + cx^{\gamma} + dx^{\delta} + \text{\&c.})^n \\ = \quad & a^n x^{n\times\alpha} + \cdots + Nx^{\pi} + \text{\&c.}; \end{aligned}$$

then will N be the chance of the happening of π in n trials.

2. Let r, s, t, &c. be the respective chances of the happening of ρ, σ, τ, &c. in one trial (B), and

$$(ax^{\alpha} + bx^{\beta} + cx^{\gamma} + dx^{\delta} + \text{\&c.})^n \times (rx^{\rho} + sx^{\sigma} + tx^{\tau} + \text{\&c.})^m$$

$$= a^n r^m x^{n\alpha + m\rho} + \cdots N' x^{\pi'} + \text{\&c.};$$

then will N' be the chance of π' happening in n trials of A, and m trials of B.

The same principles may be applied to more quantities, &c.

From some data in the *Meditat. Analyt.* and *Philos. Trans.* may be deduced other familiar propositions.

THEOREM

1.

$$\begin{aligned}
&\overline{a+b}.\overline{a+b-1}.\overline{a+b-2}.\overline{a+b-3}\ldots\overline{a+b-n+1} = \\
&\qquad a.\overline{a-1},\overline{a-2}\ldots\overline{a-n+1} \\
&\qquad + n \times a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+2} \times b \\
&\qquad + n \cdot \frac{n-1}{2} \cdot a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+3} \times b.\overline{b-1} \\
&\qquad + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+4}.b.\overline{b-1}.\overline{b-2} \\
&\qquad + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+5} \times b.\overline{b-1}.\overline{b-2}.\overline{b-3} + \cdots \\
&\qquad + L \times a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+l} \times b \times \overline{b-1}.\overline{b-2}.\overline{b-3}\ldots\overline{b-l+2}\ldots \\
&\qquad + M \times a.\overline{a-1}\ldots\overline{a-n+m} \times b \times \overline{b-1}.\overline{b-2}\ldots\overline{b-m+2} + \ldots \\
&\qquad + n \cdot \frac{n-1}{2} \cdot a.\overline{a-1}.b.\overline{b-1}.\overline{b-2}.\overline{b-3}\ldots\overline{b-n+3} \\
&\qquad + n.a \times b.\overline{b-1}.\overline{b-2}\ldots\overline{b-n+2} \\
&\qquad + b.\overline{b-1}.\overline{b-2}.\overline{b-3}\ldots\overline{b-n+1}
\end{aligned}$$

2. If for the terms $a+b-1$, $a+b-2$, $a+b-3$, &c. $a-1$, $a-2$, $a-3$, &c.; $b-1$, $b-2$, $b-3$, &c. be substituted $a+b-x$, $a+b-2x$, &c.; $a-x$, $a-2x$, &c.; $b-x$, $b-2x$, &c.; the equation resulting will be equally just: when $x=0$, it becomes the binomial theorem.

3. Let a and b be the chances of the two events, of which the sum of all the events possible or certainty is N, and after the first trial let the sum of events or certainty be diminished by 1, i.e. let it be $N-1$, and also the numbers of events to happen (a and b) or to be drawn, be diminished by 1; and in the same manner, after the second trial, let N be diminished by 2, and the number of events to happen (a and b) be still diminished by 1; that is, let it be $a-2$, if a happened the first and second time, or $a-1$ and $b-1$, if a and b happened each once only; and so on then will the chance of the event (of which the chance of happening the first time is $\frac{a+b}{N}$) happening n times together be

$$\frac{\overline{a+b}\times\overline{a+b-1}.\overline{a+b-2}\ldots\overline{a+b-n+1}}{N.\overline{N-1}.\overline{N-2}\ldots\overline{N-n+1}}=$$

$$\frac{\begin{array}{r}a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+1}+n\times a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+2}\times b\\ +n\cdot\dfrac{n-1}{2}\times a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+3}\times b.\overline{b-1}\\ +n\cdot\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+4}\times b.\overline{b-1}.\overline{b-2}\\ +\cdots L\times a.\overline{a-1}.\overline{a-2}.\overline{a-n+l}\times b.\overline{b-1}.\overline{b-2}\ldots\overline{b-l+2}\\ \ldots na.b.\overline{b-1}.\overline{b-2}.\overline{b-n+2}+b.\overline{b-1}.\overline{b-2}\ldots\overline{b-n+1}\end{array}}{N.\overline{N-1}.\overline{N-2}.\overline{N-3}.\overline{N-4}\times\ldots\overline{N-n+1}}:$$

it appears that the chance of the event's, of which the probability is $\frac{b}{a+b}$, happening $l-1$ times in (n) trials, and no more, will be

$$\frac{L\times a.\overline{a-1}.\overline{a-2}\ldots\overline{a-n+l}\times b.\overline{b-1}.\overline{b-2}\ldots\overline{b-l+2}}{\overline{a+b}.\overline{a+b-1}\ldots\overline{a+b-n+1}}.$$

From hence it follows, that every thing asserted of the former case may equally be applied to this, by only substituting for $a^l\times b^k$ in the former case the content

$$a.\overline{a-1}.\overline{a-2}\ldots\overline{a-l+l}\times b\times\overline{b-1}\times\overline{b-2}\ldots\overline{b-k+1},$$

and for N^n the content

$$N.\overline{N-1}.\overline{N-2}\ldots\overline{N-n+1}.$$

3. When more quantities a, b, c, d, &c. are contained in trials of this kind, then every thing that is asserted in the former case may also equally be applied to this, for the former will be transformed into this by writing in it for a^l, the content $a\times\overline{a-1}\times\overline{a-2}\ldots\overline{a-l+1}$, and similarly for b^k, c^p, d^q, &c. respectively, the contents $b.\overline{b-1}.\overline{b-2}\ldots\overline{b-k+1}$, $c\times\overline{c-1}.\overline{c-2}\ldots\overline{c-p+1}$, $d.\overline{d-1}.\overline{d-2}\ldots\overline{d-q+1}$:

the coefficients of the terms in both cases are the same, i.e. the coefficient of the term

$$\begin{array}{rl} & a \times \overline{a-1} \times \overline{a-2} \ldots \overline{a-l+1} \\ \times & b \times \overline{b-1}.\overline{b-2} \ldots \overline{b-k+1} \\ \times & c \times \overline{c-1}.\overline{c-2} \ldots \overline{c-p+1} \\ \times & d \times \overline{d-1}.\overline{d-2} \ldots \overline{d-q+1} \\ \times & \text{\&c.} \end{array}$$

will be

$$\begin{array}{rl} & a.\dfrac{n-1}{2} \cdot \dfrac{n-2}{3} \cdots \dfrac{n-l+1}{l} \\ \times & \overline{n-l} \times \dfrac{n-l-1}{2} \cdots \dfrac{n-l-2}{3} \cdots \dfrac{n-l-k+1}{k} \\ \times & \overline{n-l-k} \times \dfrac{n-l-k-1}{2} \times \dfrac{n-l-k-2}{k} \cdots \dfrac{n-l-k-p+1}{p} \\ \times & \overline{n-l-k-p} \times \dfrac{n-l-k-p-1}{2} \times \dfrac{n-l-k-p-2}{k} \cdots \dfrac{n-l-k-p-q+1}{q} \\ \times & \text{\&c.} \end{array}$$

in the series for

$$\begin{array}{rl} & (a+b+c+d+\text{\&c.}) \times (a+b+c+d+\text{\&c.}-1) \\ \times & (a+b+c+d+\text{\&c.}-2) \times (a+b+c+d+\text{\&c.}-3) \cdots \\ \times & (a+b+c+d+\text{\&c.}-n+1) \\ = & a \times (a-1) \cdot (a-2) \cdots (a-n+1) \\ + & n.a.(a-1) \ldots (a-n+2) \times b \\ + & n \cdot \dfrac{n-1}{2} \cdot a \cdot (a-1) \cdots (a-n+3) \times b.(b-1) + \text{\&c.} \end{array}$$

which is also the coefficient of the term $alb^k c^p d^q$ &c. in the power n of the multinomial $a+b+c+d+$ &c. expanded, i.e.

$$(a+b+c+d+\text{\&c.})^n = a^n + na^{n-1}b + n \cdot \frac{n-1}{2}\ a^{n-2}b^2 + \text{\&c.}$$

Cor. The sum of the contents of every $n-s$ of the $(n-1)$ numbers $1, 2, 3, 4, \ldots n-1 \times s \cdot \frac{s-1}{2} \cdot \frac{s-2}{3} \cdots \frac{s-r+1}{r} = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-r+1}{r}(H)\times$ sum of the contents of every $(n-s)$ of the numbers $1, 2, 3, 4, \ldots n-r-1 + H \times \frac{n-r}{r+1}(K)\times$ sum of the contents of every $(n-s-1)$ of the numbers $1, 2, 3, \ldots n-r-2\times$ sum of the numbers $1, 2, 3, \ldots r + K \times \frac{n-r-1}{r+2}(L)\times$ sum of the contents of every $(n-s-2)$ of the numbers $1, 2, 3, \ldots n-r-3\times$ sum of the products of every two of the numbers $1, 2, 3, , \ldots r, r+1, +L \times \frac{n-r-2}{r+3}(M)\times$ sum of the contents of every $(n-s-3)$ of the numbers $1, 2, 3, \ldots n-r-4\times$ sum of the contents of every three of the numbers $1, 2, 3, \ldots r, r+1, r+2+$ &c. s being less than n.