OBSERVATIONS
sur le calcul d'un Jeu de hasard*

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The game of which there is question in this Memoir, is the game of Her, on which Mr. de Montmort has treated in his *Analyse des jeux de hasard*, & on which he has had some discussions with Nicolas Bernoulli. These discussions revolved on some paradoxes which can, in that which I believe, be clarified quite easily by an exact calculation. This is that which I am going to try to do in few words, in order to show by this example that similar paradoxes do not infringe on the calculus of probabilities, & if one has sometime succeeded to obscure this calculus, as so much of other things, by some metaphysical considerations, a rigorous analysis suffices to establish it in its integrity.

§ 1. Here are the principal rules of the game which one names Her. I suppose three players, Pierre, Paul & Jacques. One draws first the places, next one decides on the one who will have the hand. Let it be Pierre who has the hand. One agrees to put into the game a certain sum; each of the players takes for this sum a certain number of tokens, & this sum reverts to the one who conserves one or many tokens, when the others have them no longer. Here is the march of the game: Pierre holds a whole deck composed of 52 cards, he gives one of them to each of the players by beginning with his right, & at the end of each coup, the one who is found to have the lowest card pays a token. Paul who is the first to the right of Pierre, has right, if he is not content with his card, to propose to exchange it with Jacques who can refuse only in the case if he has a king; in this case Jacques says *cuckcoo*. By this term, the one who has a king warns the players that his neighbor to the left having wished to undo himself of his card has been stopped by his king. It is likewise of Jacques in regard to Paul, & of Paul in regard to Pierre. It is necessary only to remark 1°. that if Pierre is not content with his card, either that it is that which he has given to himself, or that which he has been constrained to receive from Jacques, he can, having no person to whom to propose the exchange, take one of them at random among those which remain to him in the hand. 2°. That if it arrives that Pierre, having, for example, a five, does not wish to hold it & takes in the deck a jack, his card will become a jack, & thus of each other card with

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the exception of the king, because Pierre drawing a king is forced to be held to the card of which he wished to undo himself. This changing of cards being done, each player uncovers his, & the one who is found to have the lowest, by beginning with the ace, sets a token into the game. If two or more players have the same card, the one who has the primacy, that is to say, the one who is nearest to Pierre at the right, pays. Whence it follows that when one has given to the player who is at the left, a card similar to the one that one has received from him, one must hold it, as if one had given a lower one to him. The player who has lost all his tokens exits, & the others continue the game. The player who remains last, wins the pool. Here is the problem of which one demands the solution.

§ 2. Three players, Pierre, Paul & Jacques are the only ones who remain, & they have no more than one token each. Pierre holds the cards, Paul is to his right. One demands what is their lot relative to the place which they occupy, & in what proportion should the money be apportioned from the pool, if they would wish to share among them without ending the game.

§ 3. We suppose first two players only, Pierre & Paul. Let Pierre have a seven and that he wishes to change, one demands his lot. There remain 51 unknown cards, of these 51 cards, there are 31 which make him lose, namely 24 cards below seven, the three sevens & the four kings. One will have therefore the lot of Paul

$$y = \frac{31.0 + 20x}{51} = \frac{20x}{51},$$

by supposing that x is his lot if he receives a card above the seven. This card can be only an 8, 9, 10, jack or queen. If it is an eight, the lot of Pierre is $\frac{19.1+31.0}{50}$, because there remains no more than 50 cards. If it is a nine, the lot of Pierre is $\frac{15.1+35.0}{50}$. If it is a ten, the lot of Pierre is $\frac{11.1+39.0}{50}$. If it is a jack, the lot of Pierre is $\frac{7.1+43.0}{50}$. If it is a queen, the lot of Pierre is $\frac{3.1+47.0}{50}$. The lot of Pierre is therefore in general

$$\frac{19 + 15 + 11 + 7 + 3}{5.50} = \frac{55}{5.50} = \frac{11}{50}.$$

Therefore $x = \frac{39}{50}$, $y = \frac{20.39}{51.50} = \frac{780}{51.50}$.

§ 4. If Paul has a seven & if he holds it, one demands his lot by supposing that Pierre is held to the eight. There remains 51 cards, of which 24 above seven, 3 sevens, 24 below. If Pierre has one of the first 24, Paul has lost. If Pierre has one of the three sevens, let the expectation of Paul be$=z$. If Pierre has one of the 24 cards below, let the expectation of Paul be$=x$. One has, by calling y the general expectation of Paul,

$$y = \frac{24.0 + 3z + 24x}{51} = \frac{31z + 24x}{51}.$$

If Pierre has one of the three sevens, there remains 50 cards of which 24 are above, two sevens & 24 are below. Therefore 26 cards make him win & 24 make him lose, therefore

$$z = \frac{26.0 + 24}{50} = \frac{24}{50}.$$

If Pierre has one of the 24 cards below, there remain 24 cards above, 3 sevens, & 23 cards below. Therefore, because of the four kings which make Pierre lose, one has 23 cards which make Pierre win & 27 which make him lose, therefore

$$x = \frac{23.0 + 27.1}{50} = \frac{27}{50}.$$

Therefore

$$y = \frac{3.24 + 27.24}{50.51} = \frac{30.24}{50.51} = \frac{720}{50.51}.$$

If Pierre changes at the eight, one will have

$$y = \frac{24.0 + 7z + 24x}{51},$$

z & x will have the same value as above, therefore

$$y = \frac{7.24 + 24.27}{51.50} = \frac{34.24}{51.50} = \frac{816}{51.50}.$$

§ 5. If Paul has a king, he will not change; then out of the 51 cards remaining, there are only the three kings which make him lose, his expectation is therefore

$$y = \frac{3.0 + 48.1}{51} = \frac{48}{51},$$

& this expectation has place either if Pierre wishes to change or not.

§ 6. If Paul has a queen, he will not change; in this case if Pierre is held at the eight, his expectation is the following,

$$y = \frac{7.0 + 16.1 + 28z}{51}.$$

Now $z = \frac{3.0+47.1}{50}$. Therefore

$$y = \frac{16.1 + \frac{28.47}{50}}{51} = \frac{2116}{51.50}.$$

If Pierre changes at the eight, Paul has $y = \frac{7.0+12.1+32z}{51}$, $z = \frac{3.0+47.1}{50}$, therefore

$$y = \frac{12.1 + \frac{32.47}{50}}{51} = \frac{2104}{51.50}.$$

§ 7. If Paul has a jack, he will not change; in this case, if Pierre is held at the eight, one has $y = \frac{11.0+12.1+28z}{51}$, $z = \frac{7.0+43.1}{50}$, therefore

$$y = \frac{12.1 + \frac{28.43}{50}}{51} = \frac{1804}{50.51}.$$

If Pierre changes at the eight, Paul has $y = \frac{11.0+8.1+32z}{51}$, $z = \frac{7.0+43.1}{50}$, therefore

$$y = \frac{8.1 + \frac{32.43}{50}}{51} = \frac{1776}{50.51}.$$

§ 8. If Paul has a ten, he will not change; in this case, if Pierre is held at the eight, one has $y = \frac{15.0+8.1+28z}{51}$, $z = \frac{11.0+39.1}{50}$, therefore

$$y = \frac{8.1 + \frac{28.39}{50}}{51} = \frac{1492}{50.51}.$$

If Pierre changes at the eight, Paul has $y = \frac{15.0+4.1+32z}{51}$, $z = \frac{11.0+39.1}{50}$, therefore

$$y = \frac{4.1 + \frac{32.39}{50}}{51} = \frac{1428}{50.51}.$$

§ 9. If Paul has a nine, he will not change; in this case, if Pierre is held at the eight, one has $y = \frac{19.0+4.1+28z}{51}$, $z = \frac{15.0+35.1}{50}$, therefore

$$y = \frac{4.1 + \frac{28.35}{50}}{51} = \frac{1180}{50.51}.$$

If Pierre changes at the eight, Paul has $y = \frac{19.0+32z}{51}$, $z = \frac{15.0+35.1}{50}$, therefore $y = \frac{32.35}{50.51}$.

§ 10. If Paul has an eight, he will not change; in this case, if Pierre is held at the eight, one has $y = \frac{23.0+28z}{51}$, $z = \frac{19.0+31.1}{50}$, therefore $y = \frac{28.31}{50.51}$. If Pierre changes at the eight, Paul has $y = \frac{20.0+3x+28z}{51}$, $x = \frac{22.0+28.1}{50}$, $z = \frac{19.0+31.1}{50}$, therefore

$$y = \frac{20.0 + \frac{3.28}{50} + \frac{31.28}{50}}{51} = \frac{34.28}{50.51}.$$

§ 11. If Paul has a seven, & if he does not wish to change; in this case, if Pierre is held at the eight, Paul has $y = \frac{24.0+3x+24z}{51}$, $x = \frac{26.0+24.1}{50}$, $z = \frac{23.0+27.1}{50}$. Therefore

$$y = \frac{24.0 + \frac{3.24}{50} + \frac{24.27}{50}}{51} = \frac{30.24}{50.51},$$

as above. If Pierre changes at the eight, Paul has $y = \frac{24.0+7x+24z}{51}$, x & z conserving the same value, therefore

$$y = \frac{24.0 + \frac{7.24}{50} + \frac{24.27}{50}}{51} = \frac{24.34}{50.51},$$

as above. If Paul has a seven, & if he wishes to change, we have found above, $y = \frac{780}{51.50}$.

§ 12. If Paul has a six, he wishes necessarily to change. Pierre has to balance no longer; we will find, by reasoning, as above, $y = \frac{27.0+24x}{51}$,

$$x = 1 - \frac{(23+19+15+11+7+3)}{6.50} = 1 - \frac{78}{6.50} = \frac{37}{50}.$$

Therefore $y = \frac{24.37}{51.50}$.

§ 13. If Paul has a five, he wishes necessarily to change. Pierre has to balance no longer; we will have therefore $y = \frac{23.0+28x}{51}$,

$$x = 1 - \frac{(27+23+19+15+11+7+3)}{7.50} = 1 - \frac{15}{50} = \frac{35}{50}.$$

Therefore $y = \frac{28.35}{51.50}$.

§ 14. If Paul has a four, he wishes necessarily to change. Pierre has to balance no longer; we will have therefore $y = \frac{19.0+32x}{51}$,

$$x = 1 - \frac{(31+27+23+19+15+11+7+3)}{8.50} = 1 - \frac{17}{50} = \frac{33}{50}.$$

Therefore $y = \frac{32.33}{51.50}$.

§ 15. If Paul has a three, he wishes necessarily to change. Pierre has to balance no longer; we will have therefore $y = \frac{15.0+36x}{51}$,

$$x = 1 - \frac{(35+31+27+23+19+15+11+7+3)}{9.50} = 1 - \frac{19}{50} = \frac{31}{50}.$$

Therefore $y = \frac{36.31}{51.50}$.

§ 16. If Paul has a two, he wishes necessarily to change. Pierre has to balance no longer; we will have therefore $y = \frac{11.0+40x}{51}$,

$$x = 1 - \frac{(39+35+31+27+23+19+15+11+7+3)}{10.50} = 1 - \frac{21}{50} = \frac{29}{50}.$$

Therefore $y = \frac{40.29}{51.50}$.

§ 17. If Paul has an ace, he wishes necessarily to change. Pierre has to balance no longer; we will have therefore $y = \frac{7.0+44x}{51}$,

$$x = 1 - \frac{(43+39+35+31+27+23+19+15+11+7+3)}{11.50} = 1 - \frac{23}{50} = \frac{27}{50}.$$

Therefore $y = \frac{44.27}{51.50}$.

§ 18. I draw thence the following table, by reducing all the fractions to a common denominator, 25.51; the numerators will be

Lot of Paul

Cards of Paul.	Paul is held at the 7. Pierre is held at the 8.	Paul is held at the 7. Pierre changes at the 8.	Paul changes at the 7. Pierre is held at the 8.	Paul changes at the 7. Pierre changes at the 8.
King	1200	1200	1200	1200
Queen	1058	1052	1058	1052
Jack	902	888	902	888
Ten	746	724	746	724
Nine	590	560	590	560
Eight	434	476	434	476
Seven	360	408	390	390
Six	444	444	444	444
Five	490	490	490	490
Four	528	528	528	528
Three	558	558	558	558
Two	580	580	580	580
Ace	594	594	594	594
Mean	$\frac{8484}{13.25.51}$	$\frac{8502}{13.25.51}$	$\frac{8514}{13.25.51}$	$\frac{8484}{13.25.51}$

Lot of Pierre

Mean	$\frac{8091}{13.25.51}$	$\frac{8073}{13.25.51}$	$\frac{8061}{13.25.51}$	$\frac{8091}{13.25.51}$

§ 19. One can have directly the lot of Pierre. If Paul is held at the seven, here are the different lots of Pierre:

Pierre has a King	He wins necessarily
Queen	$\frac{23.1+4.0+\frac{24.3}{50}}{51}$
Jack	$\frac{19.1+8.0+\frac{24.7}{50}}{51}$
Ten	$\frac{15.1+12.0+\frac{24.11}{50}}{51}$
Nine	$\frac{11.1+16.0+\frac{24.15}{50}}{51}$
Eight if he is held	$\frac{7.1+20.0+\frac{24.19}{50}}{51}$
Eight if he changes	$\frac{4.0+\frac{4(3+7+11+15)}{50}+\frac{3.22}{50}+\frac{4.26}{50}+\frac{4(19+19+19+19+19+19)}{50}}{51}$
Seven	$\frac{4.0+\frac{4(3+7+9+11+15+19)}{50}+\frac{3.26}{50}+\frac{4(23+23+23+23+23+23)}{50}}{51}$
Six	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+\frac{23}{50}+3.1+\frac{4(27+27+27+27+27)}{50}}{51}$
Five	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+7.1+\frac{4(31+31+31+31)}{50}}{51}$
Four	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+11.1+\frac{4(35+35+35)}{50}}{51}$
Three	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+15.1+\frac{4(39+39)}{50}}{51}$
Two	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+19.1+\frac{4.43}{50}}{51}$
Ace	$\frac{4.0+\frac{4(3+7+11+15+19+23)}{50}+13.1}{51}$

If Paul changes at the seven, here are the lots of Pierre:

Pierre has a King	He wins necessarily
Queen	$\frac{19.1+4.0+\frac{28.3}{50}}{51}$
Jack	$\frac{15.1+8.0+\frac{28.7}{50}}{51}$
Ten	$\frac{11.1+12.0+\frac{28.11}{50}}{51}$
Nine	$\frac{7.1+16.0+\frac{28.15}{50}}{51}$
Eight if he is held	$\frac{3.1+20.0+\frac{28.19}{50}}{51}$
Eight if he changes	$\frac{4.0+\frac{4(3+7+11+15)}{50}+\frac{3.22}{50}+\frac{4(19+19+19+19+19+19)}{50}}{51}$
Seven	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+3.1+\frac{4(23+23+23+23+23+23)}{50}}{51}$
Six	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+7.1+\frac{4(27+27+27+27+27)}{50}}{51}$
Five	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+11.1+\frac{4(31+31+31+31)}{50}}{51}$
Four	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+15.1+\frac{4(35+35+35)}{50}}{51}$
Three	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+19.1+\frac{4(39+39)}{50}}{51}$
Two	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+23.1+\frac{4.43}{50}}{51}$
Ace	$\frac{4.0+\frac{4(3+7+11+15+19)}{50}+27.1}{51}$

§ 20. One will form thence the following table:

Lot of Pierre

Cards of Paul.	Paul is held at the 7. Pierre is held at the 8.	Paul is held at the 7. Pierre changes at the 8.	Paul changes at the 7. Pierre is held at the 8.	Paul changes at the 7. Pierre changes at the 8.
King	1275	1275	1275	1275
Queen	611	611	517	517
Jack	559	559	473	473
Ten	567	567	429	429
Nine	455	455	385	385
Eight	403	385	341	371
Seven	425	425	461	461
Six	501	501	555	555
Five	579	579	633	633
Four	641	641	695	695
Three	687	687	741	741
Two	717	717	771	771
Ace	731	731	785	785
Mean	$\frac{8091}{13.25.51}$	$\frac{8073}{13.25.51}$	$\frac{8061}{13.25.51}$	$\frac{8091}{13.25.51}$

§ 21 If Paul is held at the seven without knowing if Pierre changes at the eight, his lot will be $\frac{8493}{13.25.51}$, & the one of Pierre $\frac{8082}{13.27.51}$. If Paul changes at the seven without knowing if Pierre changes at the eight, his lot will be $\frac{8499}{13.25.51}$, & the one of Pierre $\frac{8076}{13.25.51}$. If one does not know if Paul changes at the seven, & if Pierre changes at the eight, the lot of Paul will be $\frac{8496}{13.25.51}$, & the one of Pierre $\frac{8079}{13.25.51}$.

§ 22 If Pierre is held at the eight without knowing if Paul changes at the seven, his lot will be $\frac{8079}{13.25.51}$, & the one of Paul $\frac{8499}{13.25.51}$. If Pierre changes at the eight without knowing if Paul changes at the seven, his lot will be $\frac{8082}{13.25.51}$, & the one of Paul $\frac{8493}{13.25.51}$.

§ 23. One sees thence that if Pierre is held at the eight, he agrees with Paul to change at the seven, & that if Pierre changes at the eight, he agrees with Paul to be held at the seven. Mr. de Montmort & his friends conclude thence against Nicolas Bernoulli, that this case was insoluble, because said they, if Paul knew that Pierre is held at the eight, he will change at the seven, but Pierre coming to know that Paul changes at the seven, will change at the eight, this which makes a vicious circle. But there results only thence that each will be perpetually in uncertainty on the manner of play of his adversary; consequently he will agree with Paul to change at the seven in a given coup, but he should not follow constantly this system many coups in sequence. He will agree likewise with Pierre to change at the eight in a given coup, without being able to make many coups in sequence, this which accords with the conclusions of Mr. Nicolas Bernoulli against those of Mr. de Montmort.

§ 24. We come to the solution of the problem proposed by Mr. de Montmort, in which one supposes three players, Paul, Jacques & Pierre, who have each one token. I suppose here that one does not know if they change at the seven & at the eight; we have seen that under this assumption, the lots of the two players Paul & Pierre were in ratio $8496 : 8079$. Let this ratio be as x to y. One will observe that the lots of Paul & of Jacques will be between them in this case here as the lots of Paul & of Pierre were between them in the preceding case, & that the lots of Jacques & of Pierre in this case here are between them as the lots of Paul & of Pierre. Jacques plays therefore with Paul & with Pierre, & by virtue of that which we just said, one has

$$\begin{array}{ll} \text{The probability that Jacques will gain over Paul} & = \dfrac{y}{x+y} \\ \text{The probability that Jacques will lose with Paul} & = \dfrac{x}{x+y} \\ \text{The probability that Jacques will gain over Pierre} & = \dfrac{x}{x+y} \\ \text{The probability that Jacques will lose with Pierre} & = \dfrac{y}{x+y} \end{array}$$

§ 25. One has therefore by combining these probabilities,

The probability that Jacques will gain over Paul & will lose with Pierre $= \dfrac{yy}{(x+y)^2}$

The probability that Jacques will gain over Paul & Pierre $= \dfrac{xy}{(x+y)^2}$

The probability that Jacques will lose with Paul & with Pierre $= \dfrac{xy}{(x+y)^2}$

The probability that Jacques will lose with Paul & will gain over Pierre $= \dfrac{x^2}{(x+y)^2}$

In order that Jacques exit from the game, it is necessary that he lose with Paul & with Pierre, thus the probability that Jacques will exit from the game will be $= \frac{xy}{(x+y)^2}$, therefore the probability that Jacques will remain in the game will be $= \frac{xx+xy+yy}{(x+y)^2}$. Let now the probability that Jacques will remain in the game with Paul $= \frac{A}{(x+y)^2}$, & the probability that Jacques will remain in the game with Pierre $= \frac{B}{(x+y)^2}$, one will have $A + B = xx + xy + yy$; now if Jacques loses with Paul & wins with Pierre, it is clear that it is Paul who will remain in the game with him, thus xx must enter into the value of A; if Jacques wins with Paul & loses with Pierre, it is clear that it is Pierre who will remain in the game with him, thus y must enter into the value of B. But, in the case where Jacques will win over Paul & Pierre, one knows not who will remain in the game; I make therefore $A = xx + mxy$, $B = nxy + yy$, with the condition that $m + n = 1$. One will have therefore,

The probability that Jacques will remain in the game with Paul $= \dfrac{xx+mxy}{(x+y)^2}$.

The probability that Jacques will remain in the game with Pierre $= \dfrac{nxy+yy}{(x+y)^2}$.

Actually, one has

The probability that Jacques will remain in the game with Paul & will win $= \dfrac{x^2y+mxy^2}{(x+y)^3}$.

The probability that Jacques will remain in the game with Paul & will lose $= \dfrac{x^3+mx^2y}{(x+y)^3}$.

The probability that Jacques will remain in the game with Pierre & will win $= \dfrac{nx^2y+xy^2}{(x+y)^3}$.

The probability that Jacques will remain in the game with Pierre & will lose $= \dfrac{nxy^2+y^3}{(x+y)^3}$.

The probability that Jacques will exit & that Paul will win over Pierre $= \dfrac{x^2y}{(x+y)^3}$.

The probability that Jacques will exit & that Pierre will win over Paul $= \dfrac{xy^2}{(x+y)^3}$.

Here is therefore the lots of the three players:

Lot of Paul,	Lot of Jacques,	Lot of Pierre,
$\dfrac{x^3+(m+1)x^2y}{(x+y)^3}$;	$\dfrac{(m+1)x^2y+(n+1)xy^2}{(x+y)^3}$;	$\dfrac{(n+1)xy^2+y^3}{(x+y)^3}$.

§ 26. All depends actually on the determination of the letters m & n. Now $m:n=$ the lot of Paul:lot of Pierre, by supposing that Jacques wins both of them. We examine therefore the different cases. If Jacques has an ace, the thing is impossible; if Jacques has a two, Paul can have only a two or an ace, & Pierre only an ace. If Paul has one of three twos which remain, he wins, provided that Pierre has one of the four aces, this which makes 12 cases in favor of Paul. If Paul has one of the four aces, he loses if Pierre has one of the three other aces, this which makes 12 cases in favor of Pierre. Their lots are therefore equal in this case.

§ 27. If Jacques has a three, Paul can have only a three, a two or an ace, & Pierre only a two or an ace. If Paul has one of the three threes which remain, he wins provided that Pierre has one of the four twos or one of the four aces, this which makes 24 cases in favor of Paul. If Paul has one of the four twos, he wins provided that Pierre has one of the four aces, this which makes 16 cases in favor of Paul, in all 40. Now, if Paul has a two, he loses if Pierre has one of the three other twos, this which makes 12 cases in favor of Pierre. If Paul has an ace, he loses provided that Pierre has one of the four twos or one of the three aces, this which makes 28 cases in favor of Pierre, in all 40. The lots are therefore again equal in this case; one can represent them thus;

Cases favorable to Paul $\quad 3.8+4.4=40.$

Cases favorable to Pierre $\quad 4(3+7)=40.$

§ 28. If Jacques has a four, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.12+4(4+8)=84.$

Cases favorable to Pierre $\quad 4(3+7+11)=84.$

§ 29. If Jacques has a five, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.16+4(4+8+12)=144.$

Cases favorable to Pierre $\quad 4(3+7+11+15)=144.$

§ 30. If Jacques has a six, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.20+4(4+8+12+16)=220.$

Cases favorable to Pierre $\quad 4(3+7+11+15+19)=220.$

§ 31. If Jacques has a seven, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.24+4(4+8+12+16+20)=312.$

Cases favorable to Pierre $\quad 4(3+7+11+15+19+23)=312.$

§ 32. If Jacques has an eight, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.28+4(4+8+12+16+20+24)=420.$

Cases favorable to Pierre $\quad 4(3+7+11+15+19+23+27)=420.$

§ 33. If Jacques has a nine, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.32 + 4(4 + 8 + 12 + 16 + 20 + 24 + 28) = 544.$

Cases favorable to Pierre $\quad 4(3 + 7 + 11 + 15 + 19 + 23 + 27 + 31) = 544.$

§ 34. If Jacques has a ten, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.36 + 4(4 + 8 + 12 + 16 + 20 + 24 + 28 + 32) = 684.$

Cases favorable to Pierre $\quad 4(3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35) = 684.$

§ 35. If Jacques has a jack, one will find by similar reasoning,

Cases favorable to Paul $\quad 3.40 + 4(4 + 8 + 12 + 16 + 20 + 24 + 28 + 32 + 36) = 840.$

Cases favorable to Pierre $\quad 4(3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39) = 840.$

§ 36. If Jacques has a queen, one will find by similar reasoning,

Cases favorable to Paul

$$3.44 + 4(4 + 8 + 12 + 16 + 20 + 24 + 28 + 32 + 36 + 40) = 1012.$$

Cases favorable to Pierre

$$4(3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43) = 1012.$$

§ 37. If Jacques has a king, one will find by similar reasoning,

Cases favorable to Paul

$$3.48 + 4(4 + 8 + 12 + 16 + 20 + 24 + 28 + 32 + 36 + 40 + 44) = 1200.$$

Cases favorable to Pierre

$$4(3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43 + 47) = 1200.$$

§ 38. When therefore that Jacques wins, the lots of Paul & of Pierre are equal, it is necessary therefore to make $m = n = \frac{1}{2}$, this which gives

Lot of Paul,	Lot of Jacques,	Lot of Pierre,
$\dfrac{x^3 + \frac{3}{2}x^2y}{(x+y)^3}$;	$\dfrac{\frac{3}{2}(x^2y + xy^2)}{(x+y)^3}$;	$\dfrac{\frac{3}{2}xy^2 + y^3}{(x+y)^3}$.

We will have above, $\frac{x}{x+y} = \frac{8496}{16575}$, $\frac{y}{x+y} = \frac{8079}{16575}$,

$$\begin{aligned} \log 8496 &= 3.9292145; & \log 8079 &= 3.9073576;\\ \log 16575 &= 4.2194535; & \log 16575 &= 4.2194535;\\ \log \frac{x}{x+y} &= \overline{9.7097610}; & \log \frac{y}{x+y} &= \overline{9.6879041}. \end{aligned}$$

Therefore

$\log \dfrac{x^3}{(x+y)^3}$	$= 9.1292830;$	$\log \dfrac{y^3}{(x+y)^3}$	$= 9.0637123;$
$\dfrac{x^3}{(x+y)^3}$	$= 0.134674;$	$\dfrac{y^3}{(x+y)^3}$	$= 0.115801;$
$\log \dfrac{x^2}{(x+y)^2}$	$= 9.4195220;$	$\log \dfrac{y^2}{(x+y)^2}$	$= 9.3758082;$
$\log \dfrac{y}{x+y}$	$= 9.6879041;$	$\log \dfrac{x}{x+y}$	$= 9.7097610;$
$\log \dfrac{x^2y}{(x+y)^2}$	$= 9.1074261;$	$\log \dfrac{xy^2}{(x+y)^2}$	$= 9.0855692;$
$\log \dfrac{3}{2}$	$= 0.1760913;$	$\log \dfrac{3}{2}$	$= 0.1760913;$
$\log \dfrac{\frac{3}{2}x^2y}{(x+y)^2}$	$= 9.2835174;$	$\log \dfrac{\frac{3}{2}xy^2}{(x+y)^2}$	$= 9.2616605;$
$\dfrac{\frac{3}{2}x^2y}{(x+y)^2}$	$= 0.192096;$	$\dfrac{\frac{3}{2}xy^2}{(x+y)^2}$	$= 0.182667;$
$\dfrac{x^3}{(x+y)^3}$	$= 0.134674;$	$\dfrac{\frac{3}{2}x^2y}{(x+y)^2}$	$= 0.192096;$
Lot of Paul	$= 0.326770;$	Lot of Jacques	$= 0.374763;$
$\dfrac{\frac{3}{2}xy^2}{(x+y)^2}$	$= 0.182667;$	Lot of Pierre	$= 0.298468;$
		Lot of Jacques	$= 0.374763;$
$\dfrac{y^3}{(x+y)^3}$	$= 0.115801;$	Lot of Paul	$= 0.326770;$
Lot of Pierre	$= 0.298468;$	Sum	$= 1.000001.$

§ 39. The lots of Paul, Jacques & Pierre are therefore among them as the numbers 326770, 374763, 298468, very nearly so, since the addition of the three fractions has given unity, to the last decimal nearly, on which one can not count. It will appear without doubt strange enough, that the lot of Paul is smaller than the one of Jacques. In order to understand the reason, it is necessary to observe that the lot of Paul is composed of two probabilities, that which he will remain in the game with Jacques & will win it, that which he will remain in the game with Pierre & will win it. These probabilities are $\frac{x^3+\frac{1}{2}x^2y}{(x+y)^3}$, $\frac{x^2y}{(x+y)^3}$. Likewise, the lot of Jacques is composed of two probabilities, that which he will remain in the game with Paul & will win it, that which he will remain in the game with Pierre & will win it. These probabilities are $\frac{x^2y+\frac{1}{2}xy^2}{(x+y)^3}$, $\frac{\frac{1}{2}x^2y+xy^2}{(x+y)^3}$. Now, the probability that Jacques remaining in the game, Paul will win it, is greater than the probability that Jacques remaining in the game, will win; but the probability that Jacques exiting from the game, Paul will win over Pierre, is smaller than the probability that Jacques remaining in the game, will win over Pierre, because the probability that Jacques will exit from the game is small enough, & diminishes consequently much the second probability of Paul. The calculus alone can reveal these sorts of paradoxes.