Sur les principes de la THÉORIE DES GAINS FORTUITS*

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SECOND MEMOIR
Examination of some difficulties.
§ 1.

One can wager 671 against 625, or a little more than 1 against 1, to bring forth at least one time a determined face of a cubical die in playing four coups. Jac. Bernoulli himself proposed on this subject an objection which consists in an apparent opposition which he finds between this assertion & that of the possible equality of the six faces. Because, out of 600 casts, the determined face must, at equal fortune, arrive 100 times; but it seems on the other hand that it must arrive only 75 times, because out of four casts one can wager 1 against 1 that it will not be found.

The response to this objection is evidently that in four consecutive casts where the determined face arrives, it can arrive more than one time.

Jac. Bernoulli makes a different response.

§ 2. "I put in fact, says this Geometer, that, when one plays at equal fortune, out of 600 casts the face *A* must arrive 100 times. But I deny that, if one wagers to bring forth one time the face A in four casts, there is need for this to cast the die four times. Because the 1$^{\text{st}}$, the 2$^{\text{nd}}$, the 3$^{\text{rd}}$ cast can bring forth the face *A*, in which case the rest of this quadrille of casts imputes itself in the following quadrille, so that less than 8 casts are necessary to win & lose one time. In order to make application of this remark, suppose that in all the quadrilles which make me win, the face *A* comes justly on the first cast; I will win 100 times in 100 casts: & of 600 casts, there will remain of them 500 in which *A* will be found not at all; this number being divided by 4, shows that I will lose 125 times. Now suppose that in the quadrilles which make me win, the face *A* comes constantly at the 4$^{\text{th}}$ cast; in order to win 100 times, I will employ 400 casts; & of 600 casts, there will remain only 200 of them, which divided by 4, will indicate my loss, which will be found thus equal to 50. Consequently, since in playing to four

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 30, 2009 coups, at the end of along number of games, sometimes I win, sometimes I lose; I conclude that it can be well done that one plays to a fate equal to this condition. On the contrary, if one wagered to bring forth the face *A* in three coups, in truth one would find the cases where one could by making 600 casts to win as much as lose; but in all the other cases much more numerous, one lost much more than one could win; whence it follows that one could play this game here only with loss." (*Ars. conj.* p. 26).

§ 3. *Observations on this response of J. Bernoulli.*

If it follows from the 1st hypothesis on the theory of accidental gains (Sect. I. § 7. *Mém. de 1780.* p. 436) that one supposes all the combinations completed, the principle whence the Author departs is different from the one of that theory; thus that response carries on a hypothesis different from that of the proposition which it defends; it is not therefore precisely that which the objection required. It is this that I will make better sense by an indirect refutation.

§ 4. First the comparison that the Author makes of the case where one plays to three coups to the case where one plays to four coups, is not conclusive; because there is an infinity of cases where by playing a number of coups different from the one which the calculus indicates, & by reasoning as Bernoulli does it in that response, it is found that sometimes the number of coups of loss surpass that of the coups of gain, sometimes inversely. For example, suppose a die of 10,000 faces. One can wager one against one to bring forth one time the face *A* in 6936 coups, so that out of 1,000,000 casts, if one plays at equal fortune, one can expect 1○. to bring forth 100 times the face *A*, 2○. to find as many combinations of 6936 coups offering the face *A* as of similar combinations where this face is not found at all. It is therefore here the same case as the one of the objection. And if one supposes that one recommences at each time that *A* arrives, without playing the rest of the 6936 coups, one will be able to apply to this objection the response of J. Bernoulli. Now I say that this response will be also applicable if one wagers to bring forth the face *A* in 6937 coups, or in 9998 coups, or in every other number of coups placed between these limits. It is this of which it is easy to be assured by a single test & by following step-by-step the response which I discuss.

But this argument proves simply that the consideration of the case where one plays to three coups adds no force to the reasoning. Here is one which attacks this same reasoning.

§ 5. Suppose four punters wager against one same banker on the same casts of a die, each in favor of a different face. Each of these four punters can wager one against one or a little more, to bring forth one time at least in four casts the fase for which he wagers. At the end of a great number of games, as for example, of 600, if they play to equal fortune (in order to serve myself of the expression of the Author) neither the punters, nor the banker can lose. However it is manifest that there will be no game where the banker is obliged to complete the four casts, in order to satisfy his agreements with the punters; therefore the assumption that the banker must in any case not complete the four casts, could not serve to respond to the proposed difficulty.

§ 6. *Of the case where one does not complete all the combinations.*

The manner to play admitted in the response of J. Bernoulli which I have reported, is certainly that which holds most commonly in the practice of games of chance; I have therefore believed to join here some observations on this hypothesis.

First by not playing a useless coup one economizes the time; there is a real gain for the one of the players of whom the lot is besides advantageous. But the following remark, which is independent of this, is the only one which demands any development.

§ 7. Suppose two players wager to bring forth n times the event *A* in x coups & that the one plays no useless coup, while the other will complete all the games in x coups, whether he wins or not before the last coup.

1∘. If the estimation of the lot of these two players gave for these expectations two different values, there would result a strange consequence, namely that the point of time at which one is placed in order to begin each game, is not indifferent to the player.

2∘. In order to estimate according to the principles of the calculus the lot of a player who would not complete the coups of each game, it would be necessary to suppose that he played thus out of all combinations, either by achieving each complete combination, or by making on each an equal number of games; in estimating conveniently besides those which would remain undecided.

3∘. Having made this test, I have always found that the lot of the player is not at all altered by this innovation; thus, although his lot is calculated according to another hypothesis, all the results of this calculation have appeared to me to apply equally to this here.

4∘. However it is necessary to remark that it is not exactly true that under the assumption that I just said, namely when all the conceivable combinations are supposed to take place, & when the undecided games are estimated conformably to the principles of the calculus.

§ 8. In order for me to explicate more clearly I will take a simple example. I suppose that one casts a piece marked *A*, *B*, & that one wagers to bring forth the face A one time at least in two coups. I suppose further that one makes thus m consecutive games, for example, *two* games.

In this case, if the player himself obliged to cast his piece four times in sequence although it arrives, these four casts would offer 16 conceivable combinations. And if the same player played 16 times in sequence or 16 m times, the calculus supposes that these 16 combinations would arrive each one time in the first case, m times in the second case (Sect. I. § 7).

One can therefore envision in some way the player as drawing in a lottery where the tickets would report the gain of each of these combinations. Now the mean gain of these combinations is always proportional to the number of made games; in other terms, when one makes many games, the ratio of the mean gain to the mean loss is always equal to the one which takes place when one makes only one game.

§ 9. I have doubted first that it was likewise when one did not complete the coups of each game; but I have recognized that this doubt was not based so much that one replenished the double condition to play respecting all the combinations & to estimate the undecided games according to the principles of the calculus. It is however of the very complicated cases where I know not if the suppression of the useless coups would change not at all the results. Although there is in it, if in the alleged example the 16 combinations do not return equally often; the two hypotheses, I wish to say the two ways to play without achieving or in achieving each combination, could make often the product vary. The least test on some combinations will suffice in order to convince ourselves. In the response of J. Bernoulli, for example, this author takes the extreme cases of loss & of gain, namely $1\circ$. 100 *A* successively followed by 500 other faces, $2\circ$. 100 quadrilles of tosses terminated by *A*, followed by 200 other faces (§ 2.).

If in the first case the player had accomplished the casts of each game, his gain would have been four times less & his loss the same. In the second case, his lot had not changed at all. He would not have had compensation in these extreme cases, & the player who would have achieved his four coups in each game, would not have won as much as the other.

Now 600 is not the half of the fourth power of 6; thus in 600 coups, as far as all the combinations of four casts are found, there is not found the eighth part in it. Therefore it can well be that there is no compensation between the lot of two players who would keep to this number of coups. Indeed, if one same player makes 600 coups, (supposing that he plays to equal fortune) he has good claim to expect 75 combinations where A is found & 75 where A is not found. But if he does not complete the combinations & if he recommences at each winning coup, his lot must be different.

It appears that this difference depends on two circumstances; namely 1°. of the number of homonymous combinations; $2\circ$. of the number of faces of the die, or more generally of the probability of the event at the first coup.

§ 10. Suppose that one makes m coups in achieving all the combinations in a game where the homonymous combinations are very frequent. For example, if one wagers with a cubical die marked *A*, *B*, &c. to bring forth the combinations of this form *AAAA*, *BBBB*, &c. In this case the player will make a gain inferior to the one that the calculus promised him; because if he brought forth only similar combinations, he would win only all six coups & not all two coups & likewise. One can conclude from this example that, all things equal, the more the homonymous chances are frequent, the more the player who finished the combinations plays with disadvantage.

If the player does not complete the combinations, but recommences after each gain, the frequency of the homonymous chances is equally in this case a disadvantage for him. Because he wins 4 parts out of 9, instead of 1 out of 2 which the calculus promised to him. But his disadvantage is much less than the one of the player who completes.

Here is for that which regards this first element, by joining the inverse assertions; namely, that the one who plays by achieving wins & that the one who plays without achieving loses, when the homonymous chances are more rare than the calculus supposes them: without affirming however that these gains & balances are balanced.

§ 11. See now how the number of faces of the die influence on the difference of the results of our two hypotheses.

Suppose that each face of the die is repeated m times in sequence, for example 20 times, so that the player brings forth, 20 *A*, then 20 *B* & thus in sequence. Suppose that he wins one against one by bringing forth at least one time the face A in the number of coups determined by the calculus & which depends on the number of faces of the die. If the player completes all the combinations; as much as the number of coups of each is < 20, this number influences little or even influences not at all on his lot. To the contrary, if the player recommences after each gain, it is very advantageous to him that the number of coups to make (or the number x (§ 7)) is great, & consequently it appears that we can put for principle that, if the homonymous sequences are very long & very frequent, the more faces the die will have, the more the lot of the player who recommences after each gain will have advantage.

If the number of coups to make (or x (§ 7.)) surpasses the number of coups of each homonymous sequence; if, for example, in this case each combination would have more than 20 coups, the player who completes them all would lose less than before & always less by increasing the number of the coups of each combination, up to that which finally he would win more than the calculated stake, & by increasing further he would always win.

It appears therefore that, if the number of coups of each combination is greater than the one of the homonymous sequences, or in general if the homonymous sequences are short & rare, the advantage which results from the number of the faces of the die for the player who does not complete will be insensible or null, likewise finally negative. If therefore the number of faces of the die is very small, this element has no influence; or if it has any one of them, it is in sense contrary to that which has this element when the number of faces of the die is great.

§ 12. From all this that I just said it seems to me that one can conclude that, if one plays with a die of one, two, three or four faces, to many coups, one will have from it claim to set the following thesis, which can become useful.

1∘. If the homonymous combinations are more rare in the game in question than the calculus supposes them, & if one completes all the combinations, one will win more than the calculated stake.

2∘. Inversely; if one wins more than the calculated stake by achieving all the combinations, one can conclude from it that the homonymous combinations are more rare than the calculus supposes them.

3∘. In this same game, one will win less, if one recommences after each gain.

4∘. And if one plays with a die with a considerable number of faces; when the homonymous sequences are frequent but less long than the number of coups to play, the player who does not complete all the combinations has a great advantage.

§ 13. One can infer from these four positions,

1∘. That the moment where one begins a part of game is not always indifferent when one plays many in sequence.

2∘. That, in the greater part of the games of chance that one plays in many combined coups, the player who recommences after each gain & who violates consequently the hypothesis of the calculus, will obtain however commonly a gain more near to the calculated expectation than the player who completes all the combinations.

3∘. That in general the calculated stake will be found a little deficient in practice.

§ 14. I will render account in another Memoir on some experiences made in order to verify these results. I will be content to indicate here that which Mr. d'Alembert alleges in his *Opusc. Mathém.* T. IV. p. 290. And I end with an example of the influence that interruption has on the lot of the parties under the circumstances near enough perhaps to those which occasion the greater part of ordinary games.

If one plays with the piece marked *A*, *B*, one can according to the hypothesis of the calculus wager 3 against 1 to bring forth any face *A* by playing two coups. But since in certain games it is necessary to exclude some homonymous chances, we bring together from this position & admit that out of eight coups played in sequence one will find always four *A* & four *B*. And first suppose that the player is engaged to complete each pair of two coups, whether that *A* arrive first or not.

Eight similar things four by four offer $\frac{1.2.3.4.5.6.7.8}{1.2.3.4.1.2.3.4} = 70$ combinations. By deploying them we will find 48 of them, in which the ratio of the number parts of gain to the number of parts of loss will be $= 3 : 1$; 16, in which there will be four parts of gain & no parts of loss; 6, where 2 of gain on 2 of loss. Thus the number of parts of gain will be $3.48 + 4.16 + 2.6 = 220$. The number of parts of loss will be $48 + 2.6 = 60$. So that the ratio of the number of parts of gain to the number of parts of loss will be $220 : 60 = 11 : 3$. Consequently, one can wager 11 against 3, ratio $> 3 : 1$.

If one were agreed to recommence a new game after each gain, it would be found evidently the half of all the casts which would have made each win a part, this which makes 280 parts of gain: if next one examines all the combinations, one will find 80 parts of loss, & 24 parts non terminated, which are each worth $\frac{1}{2}$, so that I will ascribe the half to gain & the other half to loss. Whence there results that the ratio of the gain to the loss will be the one of $292 : 92 = 73 : 23$.

This ratio is quite different from the first. It is approximately the calculated ratio $(3 : 1)$. And this is natural, seeing the diminution of the homonymous chances & the smallness of the original probability (§ 12.).

§ 15. These results suppose that in the games that they evaluate, one must not expect a homonymous chance longer than four consecutive A. They suppose besides that one does not take account of the more or less great frequency of the inferior homonymous chances. The 1st of these elements is rather easy to determine by experience & its variations do not complicate the calculations much; but the 2nd, that is to say the gradual diminution of the frequency of each particular chance, has two contrary disadvantages.

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I will indicate in a last Memoir on this subject some applications of the preceding reflections, & I will say a word on some difficulties of another kind.