SUR LES PROBABILITÉS*

MM. Prevost and Lhuilier

Mémoires de l'Académie royale des sciences et belles-lettres... Berlin.
Classe de mathématique, 1796, pp. 117–142.†

Problem. Let there be an urn containing some tickets of two kinds (that I will call whites & blacks), in an unknown ratio. Let a certain number of these tickets be drawn successively, without returning to the urn, at each extraction, the drawn ticket. Knowing the number of tickets of each kind which have been drawn, one demands the probability that drawing in the same manner new tickets, a given number, there will be some given number of these two kinds.

§ 1. *Lemma*. Let the first n figurate numbers be taken of any order (by taking for first order the sequence of natural numbers). Let also the first n figurate numbers be taken of another order. Let the figurate numbers of the first of these orders be multiplied, one by one by the numbers of the second, arranged in reversed order. I affirm that the sum of these products is equal to the n^{th} figurate number of an order of which the exponent is superior by one unit to the sum of the exponents of these two orders.

Symbolically. Let

$$\frac{1.2\ldots p}{1.2\ldots p}$$
$$\frac{2.3\ldots p+1}{1.2\ldots p}$$
$$\frac{3.4\ldots p+2}{1.2\ldots p}$$
$$\ldots$$
$$\frac{n.n+1\ldots p+n-1}{1.2\ldots p};$$

be the first n figurate numbers of the p^{th} order.

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 28, 2009

†Read to the Academy, 12 November 1795. This Memoir, although making part of the two Memoirs of MM. *Prevost & Lhuilier* on the *Principe par lequel on estime la probabilité des causes*, which begins the following class, must, because of the details of calculation, enter into the Classe de Mathématique. See the return of the authors, p. 16. of the Classe de Philosophie Spéculative in this volume.

Let also

$$\frac{1.2\ldots q}{1.2\ldots q}$$
$$\frac{2.3\ldots q+1}{1.2\ldots q}$$
$$\frac{3.4\ldots q+2}{1.2\ldots q}$$
$$\ldots$$
$$\frac{n.n+1\ldots q+n-1}{1.2\ldots q};$$

be the first n figurate numbers of the q^{th} order.

Let the sum of the following products be taken

$$\frac{1.2\ldots p}{1.2\ldots p} \times \frac{n.n+1\ldots q+n-1}{1.2\ldots q}$$
$$\frac{2.3\ldots p+1}{1.2\ldots p} \times \frac{n-1.n\ldots q+n-2}{1.2\ldots q}$$
$$\frac{3.4\ldots p+2}{1.2\ldots p} \times \frac{n-2.n-1\ldots q+n-3}{1.2\ldots q}$$
$$\ldots$$
$$\frac{n-1.n\ldots p+n-2}{1.2\ldots p} \times \frac{2.3\ldots q+1}{1.2\ldots q}$$
$$\frac{n.n+1\ldots p+n-1}{1.2\ldots p} \times \frac{1.2\ldots q}{1.2\ldots q}.$$

I affirm that this sum is equal to

$$\frac{n.n+1\ldots p+q+n}{1.2\ldots p+q+1};$$

namely, to the n^{th} figurate number of which the exponent of the order is $p+q+1$.

I am going to introduce by some examples the general demonstration of this proposition.

First example. By the origin of the figurate numbers, the n^{th} term of any order, is the sum of a similar number of terms of the order of which the exponent is inferior by one unit. Or else, if one multiplies the terms of any order, one by one by the sequence of units (which is the sequence of terms of the order of which the exponent is zero); the sum is the n^{th} term of an order superior by one unit to the first.

Second example. Let the second order be the one of the natural numbers.

These products are

$$\begin{array}{l}
\dfrac{n.n+1\ldots p+n-1}{1.2\ldots p}\times 1\\
\dfrac{n-1.n\ldots p+n-2}{1.2\ldots p}\times 2\\
\dfrac{n-2.n-1\ldots p+n-3}{1.2\ldots p}\times 3\\
\ldots\\
\dfrac{1.2\ldots p}{1.2\ldots p}\times n
\end{array}$$

I affirm that their sum is

$$\frac{n.n+1\ldots n+p+1}{1.2\ldots p+2};$$

namely, the n^{th} figurate number of the order of which the exponent is $p+2$.

For brevity, as much in this example as in the following: Let P', P'', P''', $\ldots P^{N-\text{II}}$, $P^{N-\text{I}}$, P^{N} be the figurate numbers of the p^{th} order. Let $\overline{P+Q}^{\text{I}}$, $\overline{P+Q}^{\text{II}}$, $\overline{P+Q}^{\text{III}}$, $\overline{P+Q}^{\text{IV}}$, $\ldots\overline{P+Q}^{N-\text{II}}$,$\overline{P+Q}^{N-\text{I}}$, $\overline{P+Q}^{N}$ be the figurate numbers of the $p+q^{\text{th}}$ order.

One has in this second example.

$$\begin{array}{llllllll}
1\times P^{N} & +2.P^{N-\text{I}} & +3P^{N-\text{II}} & +4P^{N-\text{III}} & +\cdots & & & \\
 & \cdots & +n-2P''' & +n-1P'' & +nP' & & & \\
=P^{N} & +P^{N-\text{I}} & +P^{N-\text{II}} & +P^{N-\text{III}} & +\cdots & & & \\
 & \cdots & +P''' & +P'' & +P' & =(1^{\text{st}}\text{ example}) & \overline{P+\text{I}}^{N} & \\
 & +P^{N-\text{I}} & +P^{N-\text{II}} & +P^{N-\text{III}} & +\cdots & & & \\
 & \cdots & +P''' & +P'' & +P' & \cdots & \overline{P+\text{I}}^{N-\text{I}} & \\
 & & +P^{N-\text{II}} & +P^{N-\text{III}} & +\cdots & & & \\
 & \cdots & +P''' & +P'' & +P' & \cdots & \overline{P+\text{I}}^{N-\text{II}} & \\
 & & & +P^{N-\text{III}} & +\cdots & & & \\
 & \cdots & +P''' & +P'' & +P' & \cdots & \overline{P+\text{I}}^{N-\text{III}} & \\
 & & \vdots & & \vdots & & & \\
 & & +P''' & +P'' & +P' & \cdots & \overline{P+\text{I}}^{\text{III}} & \\
 & & & +P'' & +P' & \cdots & \overline{P+\text{I}}^{\text{II}} & \\
 & & & & +P' & \cdots & \overline{P+\text{I}}^{\text{I}} & =\overline{P+\text{II}}^{N}
\end{array}$$

Third example. Let the terms of the second sequence be the triangular numbers or the figurate numbers of the second order.

$$
\begin{array}{lllllll}
1 \times P^N & +3P^{N-\mathrm{I}} & +6P^{N-\mathrm{II}} & +10P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-2.n-1}{1.2}P''' & +\frac{n-1.n}{1.2}P'' & +\frac{n-1+1}{1.2}P' & & \\
= P^N & +2P^{N-\mathrm{I}} & +3P^{N-\mathrm{II}} & +4P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +n-2P''' & +n-1P'' & +nP' & = (2^{\text{nd}} \text{ example}) & \overline{P+\mathrm{II}}^{N} \\
 & +P^{N-\mathrm{I}} & +2P^{N-\mathrm{II}} & +3P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +n-3P''' & +n-2P'' & +n-1P' & \cdots & \overline{P+\mathrm{II}}^{N-\mathrm{I}} \\
 & & +P^{N-\mathrm{II}} & +2P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +n-4P''' & +n-3P'' & +n-2P' & \cdots & \overline{P+\mathrm{II}}^{N-\mathrm{II}} \\
 & & & +P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +n-5P''' & +n-4P'' & +n-3P' & \cdots & \overline{P+\mathrm{II}}^{N-\mathrm{III}} \\
 & & \vdots & & \vdots & & \\
 & & +1P''' & +2P'' & +3P' & \cdots \overline{P+\mathrm{II}}^{\mathrm{III}} & \\
 & & & +1P'' & +2P' & \cdots \overline{P+\mathrm{II}}^{\mathrm{II}} & \\
 & & & & +P' & \cdots \overline{P+\mathrm{II}}^{\mathrm{I}} & = \overline{P+\mathrm{III}}^{N} \ (1^{\text{st}} \text{ example}).
\end{array}
$$

Fourth example. Let the terms of the second sequence be the pyramidal numbers, or figurate numbers of the third order.

$$
\begin{array}{lllllll}
1 \times P^N & +4P^{N-\mathrm{I}} & +10P^{N-\mathrm{II}} & +20P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-2.n-1.n}{1.2.3}P''' & +\frac{n-1.n.n+1}{1.2.3}P'' & +\frac{n.n+1.n+2}{1.2.3}P' & & \\
= P^N & +3P^{N-\mathrm{I}} & +6P^{N-\mathrm{II}} & +10P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-2.n-1}{1.2}P''' & +\frac{n-1.n}{1.2}P'' & +\frac{n.n+1}{1.2}P' = (1^{\text{st}} \text{ example}) & & \overline{P+\mathrm{III}}^{N} \\
 & +1P^{N-\mathrm{I}} & +3P^{N-\mathrm{II}} & +6P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-3.n-2}{1.2}P''' & +\frac{n-2.n-1}{1.2}P'' & +\frac{n-1.n}{1.2}P' & \cdots & \overline{P+\mathrm{III}}^{N-\mathrm{I}} \\
 & & +1P^{N-\mathrm{II}} & +3P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-4.n-3}{1.2}P''' & +\frac{n-3.n-2}{1.2}P'' & +\frac{n-2.n-1}{1.2}P' & \cdots & \overline{P+\mathrm{III}}^{N-\mathrm{II}} \\
 & & & +P^{N-\mathrm{III}} & +\cdots & & \\
 & \cdots & +\frac{n-5.n-4}{1.2}P''' & +\frac{n-4.n-3}{1.2}P'' & +\frac{n-3.n-2}{1.2}P' & \cdots & \overline{P+\mathrm{III}}^{N-\mathrm{III}} \\
 & & \vdots & & \vdots & & \\
 & & +1P''' & +3P'' & +6P' & \cdots & \overline{P+\mathrm{III}}^{\mathrm{III}} \\
 & & & +1P'' & +3P' & \cdots & \overline{P+\mathrm{III}}^{\mathrm{II}} \\
 & & & & +P' & \cdots & \overline{P+\mathrm{III}}^{\mathrm{I}} \\
 & & & & & & = \overline{P+\mathrm{IV}}^{N} \ (1^{\text{st}} \text{ example}),
\end{array}
$$

Generally. The proposition having been proved true for any order of terms of the second sequence, one will demonstrate (precisely in the same manner as in the preceding examples) that it is true for an order of terms of the second series of which the exponent is superior by one unit. Hence, the proposition being true for the terms of any order, & in particular for the orders of which the exponent is small, such as 1, 2, 3, . . . it is also true for all the following.

The march of the general demonstration being entirely similar to that of the preceding examples, & presenting nothing difficult, I believe I must (by motive of brevity), abstain from its development.

§ 2. *Second lemma.* Let there be two arithmetic progressions, in which the difference of the successive terms is the same. Let be taken the products of m dimensions composed in the following manner:

$1°$. The continual product of the first m terms of the first progression.

$2°$. The continual product of the first $m-1$ terms of the first progression, by the first term of the second.

$3°$. The continual product of the first $m-2$ terms of the first progression, by the one of the first two terms of the second.

$4°$. The continual product of the first $m-3$ terms of the first progression, by the one of the first three terms of the second.

$5°$. The continual product of the first $m-4$ terms of the first progression, by the one of the first four terms of the second.

And thus in sequence, by diminishing always by one unit the number of the terms of the first sequence taken for factors, & by substituting in the factor subtracted from this sequence, the following term from the second, until one arrives to the continual product of the first m terms of the second sequence.

Let next these continual products be multiplied one by one, by the successive coefficients of the terms of the binomial of which the exponent is m.

I affirm, that the sum of all these products, is equal to the continual product of the terms in arithmetic progression, of which the difference is the same as that of the first two, of which the number of terms is also the same, & of which the first term is the sum of the first two terms of these progressions.

Symbolically. Let

$$\begin{array}{cccccc} p, & p+d, & p+2d, & p+3d, & \ldots & p+m-1d, \\ q, & q+d, & q+2d, & q+3d, & \ldots & q+m-1d, \end{array}$$

be two arithmetic progressions, of which the first terms are p & q, in each of which the difference of two successive terms is d, & of which the number of terms is m. Let the sum of the continual products be taken.

$$1(p.p+d.p+2d.p+3d.p+4d\ldots$$
$$\ldots p+m-4d.p+m-3d.p+m-2d.p+m-1d)$$
$$\frac{m}{1}(p.p+d.p+2d.p+3d.p+4d\ldots$$
$$\ldots p+m-4d.p+m-3d.p+m-2d..q)$$
$$\frac{m}{1}\cdot\frac{m-1}{2}(p.p+d.p+2d.p+3d.p+4d\ldots$$
$$\ldots p+m-4d.p+m-3d.q.q+d)$$

$$\frac{m}{1}\cdots\frac{m-2}{3}(p.p+d.p+2d.p+3d.p+4d\ldots$$
$$\ldots p+m-4d.q.q+d.q+2d)$$
$$\frac{m}{1}\cdots\frac{m-3}{4}(p.p+d.p+2d.p+3d.p+4d\ldots$$
$$\ldots q.q+d.q+2d.q+3d)$$

$$\vdots$$

$$\frac{m}{1}\cdots\frac{m-3}{4}(p.p+d.p+2d.p+3d.q\ldots$$
$$\ldots q+m-8d.q+m-7d.q+m-6d.q+m-5d)$$
$$\frac{m}{1}\cdot\frac{m-2}{3}(p.p+d.p+2d.q.q+d\ldots$$
$$\ldots q+m-7d.q+m-6d.q+m-5d.q+m-4d)$$
$$\frac{m}{1}\cdot\frac{m-1}{2}(p.p+d.q.q+d.q+2d\ldots$$
$$\ldots q+m-6d.q+m-5d.q+m-4d.q+m-3d)$$
$$\frac{m}{1}(p.q.q+d.q+2d.q+3d\ldots$$
$$\ldots q+m-5d.q+m-4d.q+m-3d.q+m-2d)$$
$$1(q.q+d.q+2d.q+3d.q+4d\ldots$$
$$\ldots q+m-4d.q+m-3d.q+m-2d.q+m-1d)$$

I affirm, that the sum of all these products is equal to the following continual product:

$$p+q.p+q+d.p+q+2d.p+q+3d.p+q+4d\ldots$$
$$\ldots p+q+m-4d.p+q+m-3d.p+q+m-2d.p+q+m-1d.$$

I am going to introduce next through some examples the general demonstration.

First example. Let each product be a single factor.

$$\begin{array}{l}1.p\\+1.q\end{array}=p+q$$

Second example. Let each product be composed of two factors, or let $m=2$.

$$\begin{array}{l}1.p.p+d\\+2.p.q\\+1.q.q+d\end{array}=\begin{array}{l}p.p+d+p.q\\\qquad+p.q+q.q+d\end{array}=\begin{array}{l}p(p+q+d)\\+q(p+q+d)\end{array}=p+q.p+q+d\ (1^{\text{st}}\text{ example}).$$

Third example. Let each product be composed of three factors, or $m=3$.

$$\begin{array}{l}1.p.p+d.p+2d\\+3.p.p+d.q\\+3.p.q.q+d\\+1.q.q+d.q+2d\end{array}=\begin{array}{l}1(p.p+d.p+2d+p.p+d.q\\\qquad+2(p.p+d.q+p.q.q+d\\\qquad+1(p.q.q+d+q.q+d.q+2d)\end{array}$$

$$= p + q + 2d \begin{cases} p.p + d \\ + 2.p.q \\ + q.q + d \end{cases} = p + q + 2d.p + q.p + q + d \, (2^{\text{nd}} \text{ example})$$

$$= p + q.p + q + d.p + q + 2d$$

Fourth example. Let each product be composed of four factors, or let $m = 4$.

$$\begin{aligned}
&1p.p + d.p + 2d.p + 3d \\
&+ 4.p.p + d.p + 2d.q \\
&+ 6.p.p + d.q.q + d \\
&+ 4p.q.q + d.q + 2d \\
&+ 1.q.q + d.q + 2d.q + 3d \\
&= 1(p.p + d.p + 2d.p + 3d + p.p + d.p + 2d.q \\
&\quad + 3(p.p + d.p + 2d.q + p.p + d.q.q + d) \\
&\qquad + 3(p.p + d.q.q + d + p.q.q + d.q + 2d) \\
&\qquad\quad + 1(p.q.q + d.q + 2d + q.q + d.q + 2d.q + 3d)
\end{aligned}$$

$$= p + q + 3d \begin{cases} p.p + d.p + 2d \\ + 3.p.p + d.q \\ + 3.p.q.q + d \\ + 1.q.q + d.q + 2d \end{cases} = p + q + 3d.p + q.p + q + d.p + q + 2d$$

(3rd example) $= p + q.p + q + d.p + q + 2d.p + q + 3d$.

Fifth example. Let each product be composed of five factors, or let $m = 5$.

$$\begin{aligned}
&1p.p + d.p + 2d.p + 3d.p + 4d \\
&+ 5.p.p + d.p + 2d.p + 3d.q \\
&+ 10.p.p + d.p + 2d.q.q + d \\
&+ 6.p.p + d.q.q + d.q + 2d \\
&+ 5p.q.q + d.q + 2d.q + 3d \\
&+ 1.q.q + d.q + 2d.q + 3d.q + 4d \\
&= 1(p.p + d \ldots p + 4d + p \ldots p + 3d.q \\
&\quad 4(p \ldots p + 3d.q + p \ldots p + 2d.q.q + d) \\
&\qquad + 6(p \ldots p + 2d.q.q + d + p.p + d.q \ldots q + 2d) \\
&\qquad\quad + 4(p.q \ldots q + 2d + q \ldots q + 3d) \\
&+ 1(p.q \ldots q + 3d + q \ldots q + 4d)
\end{aligned}$$

$$= p{+}q{+}4d \begin{cases} p.p + d.p + 2d.p + 3d \\ +4.p.p + d.p + 2d.q \\ + 6.p.p + d.q.q + d \\ + 4.p.q + d.q + 2d \\ + 1.q.q + d.q + 2d.q + 3d \end{cases} = p + q + 4d.p + q.p + q + d \ldots p + q + 2d.$$

$p + q + 3d$ (4[th] example) $= p + q . p + q + d . p + q + 2d \ldots p + q + 3d.$
$p + q + 4d.$

Generally. The proportion having been proved true for a certain exponent m; one demonstrates, precisely in the same manner, that it is also true for an exponent greater than unity. Hence, the proportion having been demonstrated true for the small values of m, such as 1, 2, 3, 4, 5, $\ldots$ it is true for all the successive values of m (whole & positive).

I suppress the general development which has no difficulty being entirely conformed to the one of the preceding examples.

§ 3. *Etiological principle.* If an event can be produced by a number n of different causes, the probabilities of the existence of these causes taken from the event, are among them as the probability of the event taken from these causes. And (consequently) the probability of the existence of each of them is equal to the probability of the event taken from this cause, divided by the sum of all the probabilities of the event taken from each of these causes.

Mr. de la Place is the first who has enunciated in a precise manner this principle, second in consequences. (See the Mémoires des Sçavans étrangers, T. VI. See also Classe de Philos. from this Vol. p. 8)

§ 4. *Problem.* Let there be an urn containing a number n of tickets; one has drawn $p + q$ tickets, of which p are white & q are nonwhite (that I will call black). One demands the probabilities that the white tickets & the black tickets from the urn were some given numbers, under the assumption that at each drawing one has not returned into the urn the drawn ticket.

The original composition of the urn, as for the numbers of the white & black tickets, can be each of the following:

Number of white tickets.	Number of black tickets.
$n - q$	q
$n - (q + 1)$	$q + 1$
$n - (q + 2)$	$q + 2$
$n - (q + 3)$	$q + 3$
$\vdots$	$\vdots$
$p + 3$	$n - (p + 3)$
$p + 2$	$n - (p + 2)$
$p + 1$	$n - (p + 1)$
p	$n - p$

The corresponding probabilities to draw (in a determined order) p white tickets & q black tickets, conforming to the assumption that the drawn tickets are not returned into the urn, are such that it follows:

$$\frac{n-q}{n}\cdot\frac{n-(q+1)}{n-1}\cdot\frac{n-(q+2)}{n-2}\cdots\frac{n-(q+p-2)}{n-(p-2)}\cdot\frac{n-(q+p-1)}{n-(p-1)}$$
$$\times\frac{q}{n-p}\cdot\frac{q-1}{n-(p+1)}\cdot\frac{q-2}{n-(p+2)}\cdot\frac{q-3}{n-(p+3)}\cdots$$
$$\cdots\frac{3}{n-(p+q-3)}\cdot\frac{2}{n-(p+q-2)}\cdot\frac{1}{n-(p+q-1)}.$$
$$\frac{n-(q+1)}{n}\cdot\frac{n-(q+2)}{n-1}\cdot\frac{n-(q+3)}{n-2}\cdots\frac{n-(q+p-1)}{n-(p-2)}\cdot\frac{n-(q+p)}{n-(p-1)}$$
$$\times\frac{q+1}{n-p}\cdot\frac{q}{n-(p+1)}\cdot\frac{q-1}{n-(p+2)}\cdot\frac{q-2}{n-(p+3)}\cdots$$
$$\cdots\frac{4}{n-(p+q-3)}\cdot\frac{3}{n-(p+q-2)}\cdot\frac{2}{n-(p+q-1)}.$$
$$\frac{n-(q+2)}{n}\cdot\frac{n-(q+3)}{n-1}\cdot\frac{n-(q+4)}{n-2}\cdots\frac{n-(q+p)}{n-(p-2)}\cdot\frac{n-(q+p+1)}{n-(p-1)}$$
$$\times\frac{q+2}{n-p}\cdot\frac{q+1}{n-(p+1)}\cdot\frac{q}{n-(p+2)}\cdot\frac{q-1}{n-(p+3)}\cdots$$
$$\cdots\frac{5}{n-(p+q-3)}\cdot\frac{4}{n-(p+q-2)}\cdot\frac{3}{n-(p+q-1)}.$$
$$\frac{n-(q+3)}{n}\cdot\frac{n-(q+4)}{n-1}\cdot\frac{n-(q+5)}{n-2}\cdots\frac{n-(q+p+1)}{n-(p-2)}\cdot\frac{n-(q+p+2)}{n-(p-1)}$$
$$\times\frac{q+3}{n-p}\cdot\frac{q+2}{n-(p+1)}\cdot\frac{q+1}{n-(p+2)}\cdot\frac{q}{n-(p+3)}\cdots$$
$$\cdots\frac{6}{n-(p+q-3)}\cdot\frac{5}{n-(p+q-2)}\cdot\frac{4}{n-(p+q-1)}.$$
$$\vdots$$
$$\frac{p+3}{n}\cdot\frac{p+2}{n-1}\cdot\frac{p+1}{n-2}\cdots\frac{5}{n-(p-2)}\cdot\frac{4}{n-(p-1)}$$
$$\times\frac{n-(p+3)}{n-p}\cdot\frac{n-(p+4)}{n-(p+1)}\cdot\frac{n-(p+5)}{n-(p+2)}\cdot\frac{n-(p+6)}{n-(p+3)}\cdots$$
$$\cdots\frac{n-(p+q)}{n-(p+q-3)}\cdot\frac{n-(p+q+1)}{n-(p+q-2)}\cdot\frac{n-(p+q+2)}{n-(p+q-1)}.$$
$$\frac{p+2}{n}\cdot\frac{p+1}{n-1}\cdot\frac{p}{n-2}\cdots\frac{4}{n-(p-2)}\cdot\frac{3}{n-(p-1)}$$
$$\times\frac{n-(p+2)}{n-p}\cdot\frac{n-(p+3)}{n-(p+1)}\cdot\frac{n-(p+4)}{n-(p+2)}\cdot\frac{n-(p+5)}{n-(p+3)}\cdots$$
$$\cdots\frac{n-(p+q-1)}{n-(p+q-3)}\cdot\frac{n-(p+q)}{n-(p+q-2)}\cdot\frac{n-(p+q+1)}{n-(p+q-1)}.$$

$$\frac{p+1}{n}\cdot\frac{p}{n-1}\cdot\frac{p-1}{n-2}\cdots\frac{3}{n-(p-2)}\cdot\frac{2}{n-(p-1)}$$
$$\times\frac{n-(p+1)}{n-p}\cdot\frac{n-(p+2)}{n-(p+1)}\cdot\frac{n-(p+3)}{n-(p+2)}\cdot\frac{n-(p+4)}{n-(p+3)}\cdots$$
$$\cdots\frac{n-(p+q-2)}{n-(p+q-3)}\cdot\frac{n-(p+q-1)}{n-(p+q-2)}\cdot\frac{n-(p+q)}{n-(p+q-1)}.$$
$$\frac{p}{n}\cdot\frac{p-1}{n-1}\cdot\frac{p}{n-2}\cdots\frac{2}{n-(p-2)}\cdot\frac{1}{n-(p-1)}$$
$$\times\frac{n-p}{n-p}\cdot\frac{n-(p+1)}{n-(p+1)}\cdot\frac{n-(p+2)}{n-(p+2)}\cdot\frac{n-(p+3)}{n-(p+3)}\cdots$$
$$\frac{n-(p+q-3)}{n-(p+q-3)}\cdot\frac{n-(p+q-2)}{n-(p+q-2)}\cdot\frac{n-(p+q-1)}{n-(p+q-1)}.$$

As all these terms are divided by the same continual product, $n.n-1.n-2\ldots n-(p+q-1)$, these probabilities are among them as the numerators of the fractions which express them; & also as these numerators divided by the same continual product; $1.2.3\ldots q-1.q$.

Therefore also (§ 3); the probabilities of the supposed compositions of the urn, are among them respectively as the quotients which one obtains by these divisions. And the absolute probabilities of these compositions are respectively these quotients divided by their sum.

Now (§ 1) the sum of these quotients is

$$\frac{n-(p+q-1).n-(p+q-2).n-(p+q-3)\ldots n-1.n.n+1}{1.2.3\ldots p+q-1.p+q.p+q+1}$$

Therefore (§ 3), the probabilities of the supposed compositions of the urn are respectively such that it follows.

$$\frac{1.2.3\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$

$$\begin{cases} \frac{n-q}{1}\cdot\frac{n-(q+1)}{2}\cdots\frac{n-(p+q-1)}{p}\times\frac{q}{1}\cdot\frac{q-1}{2}\cdots\frac{1}{q} \\ \frac{n-(q+1)}{2}\cdot\frac{n-(q+2)}{2}\cdots\frac{n-(p+q)}{p}\times\frac{q+1}{1}\cdot\frac{q}{2}\cdots\frac{2}{q} \\ \frac{n-(q+2)}{2}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+1)}{p}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{3}{q} \\ \frac{n-(q+3)}{2}\cdot\frac{n-(q+4)}{2}\cdots\frac{n-(p+q+2)}{p}\times\frac{q+3}{1}\cdot\frac{q+2}{2}\cdots\frac{4}{q} \\ \vdots \\ \frac{p+3}{1}\cdot\frac{p+2}{2}\cdots\frac{4}{p}\times\frac{n-(p+3)}{1}\cdot\frac{n-(p+4)}{2}\cdots\frac{n-(p+q+2)}{q} \\ \frac{p+2}{1}\cdot\frac{p+1}{2}\cdots\frac{3}{p}\times\frac{n-(p+2)}{1}\cdot\frac{n-(p+3)}{2}\cdots\frac{n-(p+q+1)}{q} \\ \frac{p+1}{1}\cdot\frac{p}{2}\cdots\frac{2}{p}\times\frac{n-(p+1)}{1}\cdot\frac{n-(p+2)}{2}\cdots\frac{n-(p+q)}{q} \\ \frac{p}{1}\cdot\frac{p-1}{2}\cdots\frac{1}{p}\times\frac{n-p}{1}\cdot\frac{n-(p+1)}{2}\cdots\frac{n-(p+q-1)}{q} \end{cases}$$

§ 5. *Problem.* One demands that of these assumptions which is the most probable.

Let $\begin{matrix} p+z, \\ n-(p+z) \end{matrix}$ & $\begin{matrix} p+z-1, \\ n-(p+z-1) \end{matrix}$ be two successive supposed numbers of $\begin{matrix}\text{white}\\ \text{black}\end{matrix}$ tickets.

The corresponding probabilities of these assumptions are between them as

$$p+z.p+z-1.p+z-2\ldots z+1\times n-(p+z).n-(p+z+1)\ldots n-(p+q+z-1)$$
$$\&\ p+z-1.p+z-2\ldots z+1.z\times n-(p+z-1)n-(p+z)\ldots n-(p+q+z-2).$$

Therefore, these probabilities are between them in the ratio of

$$p+z.n-(p+q+z-1) \text{ to } z.n-(p+z-1).$$

But, one of these probabilities being the greatest, this ratio approaches most in the ratio of equality; hence, in the case of the *maximum*, one has very nearly the equation

$$p+z.n-(p+q+z-1)=z.n-(p+z-1);$$

or

$$p+z:z=n-(p+z-1):n-(p+q+z-1);$$

whence one deduces,

$$p+z:p=n-(p+z-1):q,$$

or,

$$p+2:n-(p+z-1)=p:q.$$

Hence, in the case of the *maximum*, the number of white tickets is to the number of black tickets (increased by one unit, which is negligible so small as the total number of tickets is great), in the ratio of the extracted white tickets, to the extracted black tickets.

§ 6. *Problem.* All being posed as in § 4. One demands the probabilities to bring forth in a given number r of new drawings made in the same manner, some given numbers $r - m$, & m of white & black tickets.

Principle of the solution. The probabilities of the demanded events, corresponding to the assumptions of these causes, are in ratio composed of the probabilities of these causes, & of the probabilities of the event depending on these causes. And the probability of the event is the sum of these probabilities.

I am going to introduce to the general solution through some particular examples, relative to the particular values of m.

1°. Seek the probability that the r tickets newly extracted are white.

The probabilities of this event, corresponding to the last r hypotheses on the composition of the urn, vanishes.

The probabilities of the event, corresponding to the remaining hypotheses, are such as it follows.

$$\frac{1.2.p+q+1}{n-(p+q-1).n-(p+q-2)\dots n+1}$$

$$
\left\{
\begin{aligned}
&\frac{n-q}{1}\cdot\frac{n-(q+1)}{2}\cdots\frac{n-(p+q-1)}{p}\times\frac{q}{1}\cdot\frac{q-1}{2}\cdots\frac{1}{q}\\
&\qquad\times\frac{n-(p+q)}{n-(p+q)}\cdot\frac{n-(p+q+1)}{n-(p+q+1)}\cdots\frac{n-(p+q+r-1)}{n-(p+q+r-1)}\\
&\frac{n-(q+1)}{1}\cdot\frac{n-(q+2)}{2}\cdots\frac{n-(p+q)}{p}\times\frac{q+1}{1}\cdot\frac{q}{2}\cdots\frac{2}{q}\\
&\qquad\times\frac{n-(p+q+1)}{n-(p+q)}\cdot\frac{n-(p+q+2)}{n-(p+q+1)}\cdots\frac{n-(p+q+r)}{n-(p+q+r-1)}\\
&\frac{n-(q+2)}{1}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+1)}{p}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{3}{q}\\
&\qquad\times\frac{n-(p+q+2)}{n-(p+q)}\cdot\frac{n-(p+q+3)}{n-(p+q+1)}\cdots\frac{n-(p+q+r+1)}{n-(p+q+r-1)}\\
&\qquad\vdots\\
&\frac{p+r+2}{1}\cdot\frac{p+r+1}{2}\cdots\frac{r+3}{p}\times\frac{n-(p+r+2)}{1}\cdot\frac{n-(p+r+1)}{2}\cdots\frac{n-(p+q+r+1)}{q}\\
&\qquad\times\frac{r+2}{n-(p+q)}\cdot\frac{r+1}{n-(p+q+1)}\cdots\frac{3}{n-(p+q+r-1)}\\
&\frac{p+r+1}{1}\cdot\frac{p+r}{2}\cdots\frac{r+2}{p}\times\frac{n-(p+r+1)}{1}\cdot\frac{n-(p+r)}{2}\cdots\frac{n-(p+q+r)}{q}\\
&\qquad\times\frac{r+1}{n-(p+q)}\cdot\frac{r}{n-(p+q+1)}\cdots\frac{2}{n-(p+q+r-1)}\\
&\frac{p+r}{1}\cdot\frac{p+r-1}{2}\cdots\frac{r+1}{p}\times\frac{n-(p+r)}{1}\cdot\frac{n-(p+r-1)}{2}\cdots\frac{n-(p+q+r-1)}{q}\\
&\qquad\times\frac{r}{n-(p+q)}\cdot\frac{r-1}{n-(p+q+1)}\cdots\frac{1}{n-(p+q+r-1)}.
\end{aligned}
\right.
$$

$$
\begin{aligned}
&=\frac{p+1}{n-(p+q+r-1)}\cdot\frac{p+2}{n-(p+q+r-2)}\cdots\frac{p+r}{n-(p+q-1)}\\
&\qquad\times\frac{1}{n-(p+q-1)}\cdot\frac{2}{n-(p+q-2)}\cdots\frac{p+q+1}{n+1}
\end{aligned}
$$

$$\begin{cases} \frac{n-q}{1}\cdot\frac{n-(q+1)}{2}\cdots\frac{n-(p+q+r-1)}{p+r}\times\frac{q}{1}\cdot\frac{q-1}{2}\cdots\frac{1}{q} \\ \frac{n-(q+1)}{1}\cdot\frac{n-(q+2)}{2}\cdots\frac{n-(p+q+r)}{p+r}\times\frac{q+1}{1}\cdot\frac{q}{2}\cdots\frac{2}{q} \\ \frac{n-(q+2)}{1}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+r+1)}{p+r}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{3}{q} \\ \vdots \\ \frac{p+r+2}{1}\cdot\frac{p+r+1}{2}\cdots\frac{3}{p+r}\times\frac{n-(p+r+2)}{1}\cdot\frac{n-(p+r+3)}{2}\cdots\frac{n-(p+q+r+1)}{q} \\ \frac{p+r+1}{1}\cdot\frac{p+r}{2}\cdots\frac{2}{p+r}\times\frac{n-(p+r+1)}{1}\cdot\frac{n-(p+r+2)}{2}\cdots\frac{n-(p+q+r)}{q} \\ \frac{p+r}{1}\cdot\frac{p+r-1}{2}\cdots\frac{1}{p+r}\times\frac{n-(p+r)}{1}\cdot\frac{n-(p+r+1)}{2}\cdots\frac{n-(p+q+r-1)}{q} \end{cases}$$

And the probability of the event is the sum of these last quantities; namely (§ 1).

$$\begin{aligned} &\frac{p+1}{n-(p+q+r+1)}\cdot\frac{p+2}{n-(p+q+r-2)}\cdots\frac{p+r}{n-(p+q-1)} \\ &\quad\times\frac{1}{n-(p+q-1)}\cdot\frac{2}{n-(p+q-2)}\cdots\frac{p+q+1}{n+1} \\ &\quad\times\frac{n-(p+q+r-1)}{1}\cdot\frac{n-(p+q+r-2)}{2}\cdots\frac{n+1}{p+q+r+1} \\ =&\frac{p+1.p+2\ldots p+r}{p+q+2.p+q+3\ldots p+q+r+1}. \end{aligned}$$

Likewise, the probability that the r tickets newly extracted will be black, is

$$\frac{q+1.q+2\ldots q+r}{p+q+2.p+q+3\ldots p+q+r+1}.$$

2°. Seek the probability that of the r tickets newly extracted, there are $r-1$ white and 1 black.

The probabilities of this event corresponding to the first & to the $r-1$ last hypotheses on the composition of the urn, vanish.

The probabilities of the event, (in a given order) corresponding to the remaining hypotheses, are respectively the products of the quantity

$$\frac{1}{n-(p+q-1)}\cdot\frac{2}{n-(p+q-2)}\cdots\frac{p+q+1}{n+1},$$

by the following quantities.

$$\begin{aligned} &\frac{(n-(q+1)}{1}\cdot\frac{n-(q+2)}{2}\cdots\frac{n-(p+q)}{p}\times\frac{q+1}{1}\cdot\frac{q}{2}\cdots\frac{2}{q} \\ &\quad\times\frac{n-(p+q+1)}{n-(p+q)}\cdot\frac{n-(p+q+2)}{n-(p+q+1)}\cdots\frac{n-(p+q+r-1)}{n-(p+q+r-2)}\times\frac{1}{n-(p+q+r-1)} \end{aligned}$$

$$\frac{(n-(q+2)}{1}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+1)}{p}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{3}{q}$$
$$\times\frac{n-(p+q+2)}{n-(p+q)}\cdot\frac{n-(p+q+3)}{n-(p+q+1)}\cdots\frac{n-(p+q+r)}{n-(p+q+r-2)}\times\frac{2}{n-(p+q+r-1)}$$
$$\frac{(n-(q+3)}{1}\cdot\frac{n-(q+4)}{2}\cdots\frac{n-(p+q+2)}{p}\times\frac{q+3}{1}\cdot\frac{q+2}{2}\cdots\frac{4}{q}$$
$$\times\frac{n-(p+q+3)}{n-(p+q)}\cdot\frac{n-(p+q+4)}{n-(p+q+1)}\cdots\frac{n-(p+q+r+1)}{n-(p+q+r-2)}\times\frac{3}{n-(p+q+r-1)}$$

$\vdots$

$$\frac{p+r+1}{1}\cdot\frac{p+r}{2}\cdots\frac{r+2}{p}\times\frac{n-(p+r+1)}{1}\cdot\frac{n-(p+r+2)}{2}\cdots\frac{n-(p+q+r)}{q}$$
$$\times\frac{r+1}{n-(p+q)}\cdot\frac{r}{n-(p+q+1)}\cdots\frac{3}{n-(p+q+r-2)}\times\frac{n-(p+q+r+1)}{n-(p+q+r-1)}$$
$$\frac{p+r}{1}\cdot\frac{p+r-1}{2}\cdots\frac{r+1}{p}\times\frac{n-(p+r)}{1}\cdot\frac{n-(p+r+1)}{2}\cdots\frac{n-(p+q+r-1)}{q}$$
$$\times\frac{r}{n-(p+q)}\cdot\frac{r-1}{n-(p+q+1)}\cdots\frac{2}{n-(p+q+r-2)}\times\frac{n-(p+q+r)}{n-(p+q+r-1)}$$
$$\frac{p+r-1}{1}\cdot\frac{p+r-2}{2}\cdots\frac{r}{p}\times\frac{n-(p+r-1)}{1}\cdot\frac{n-(p+r)}{2}\cdots\frac{n-(p+q+r-2)}{q}$$
$$\times\frac{r-1}{n-(p+q)}\cdot\frac{r-2}{n-(p+q+1)}\cdots\frac{1}{n-(p+q+r-2)}\times\frac{n-(p+q+r-1)}{n-(p+q+r-1)}$$

$$=\frac{p+1.p+2\ldots p+r-1.q+1}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}$$
$$\times\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$

$$\begin{cases}
\frac{n-(q+1)}{1}\cdot\frac{n-(q+2)}{2}\cdots\frac{n-(p+q+r-1)}{p+r-1}\times\frac{q+1}{1}\cdot\frac{q}{2}\cdots\frac{1}{q+1}\\
\frac{n-(q+2)}{1}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+r)}{p+r-1}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{2}{q+1}\\
\frac{n-(q+3)}{1}\cdot\frac{n-(q+4)}{2}\cdots\frac{n-(p+q+r+1)}{p+r-1}\times\frac{q+3}{1}\cdot\frac{q+2}{2}\cdots\frac{3}{q+1}\\
\vdots\\
\frac{p+r+1}{1}\cdot\frac{p+r}{2}\cdots\frac{3}{p+r-1}\times\frac{n-(p+q+r+1)}{1}\cdot\frac{n-(p+q+r)}{2}\cdots\frac{n-(p+r+1)}{q+1}\\
\frac{p+r}{1}\cdot\frac{p+r-1}{2}\cdots\frac{2}{p+r-1}\times\frac{n-(p+q+r)}{1}\cdot\frac{n-(p+q+r-1)}{2}\cdots\frac{n-(p+r)}{q+1}\\
\frac{p+r-1}{1}\cdot\frac{p+r-2}{2}\cdots\frac{1}{p+r-1}\times\frac{n-(p+q+r-1)}{1}\cdot\frac{n-(p+q+r-2)}{2}\cdots\frac{n-(p+r-1)}{q+1}
\end{cases}$$

And the probability of the event is the sum of these last quantities; namely (§ 1)

$$\frac{p+1.p+2\ldots p+r-1.q+1}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}$$
$$\times\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$
$$\times\frac{n-(p+q+r-1).n-(p+q+r-2)\ldots q+1}{1.2\ldots p+q+r+1}$$
$$=\frac{p+1.p+2\ldots p+r-1.q+1}{p+q+2.p+q+3\ldots p+q+r+1}.$$

The probability to bring forth in any order, $r-1$ white & 1 black, is

$$r\times\frac{p+1.p+2\ldots p+r-1.q+1}{p+q+2.p+q+3\ldots p+q+r+1}$$

The probability to bring forth in any order, 1 white & $r-1$ black, is

$$r\times\frac{q+1.q+2\ldots q+r-1.p+1}{p+q+2.p+q+3\ldots p+q+r+1}.$$

3°. Seek the probability that of the r extracted tickets, there will be of them, $r-2$ white and two black.

The probabilities of this event corresponding to the first two & to the last $r-2$ hypotheses on the composition of the urn, vanish.

The probability of the event (in a given order), correspond to the remaining hypotheses, are respectively the products of the quantity

$$\frac{1}{n-(p+q-1)}\cdot\frac{2}{n-(p+q-2)}\cdots\frac{p+q+1}{n+1},$$

by the following quantities.

$$\frac{(n-(q+2)}{1}\cdot\frac{n-(q+3)}{2}\cdots\frac{n-(p+q+1)}{p}\times\frac{q+2}{1}\cdot\frac{q+1}{2}\cdots\frac{3}{q}\times\frac{n-(p+q+2)}{n-(p+q)}\cdot$$
$$\frac{n-(p+q+3)}{n-(p+q+1)}\cdots\frac{n-(p+q+r-1)}{n-(p+q+r-3)}\times\frac{2}{n-(p+q+r-2)}\cdot\frac{1}{n-(p+q+r-1)}$$
$$\frac{(n-(q+3)}{1}\cdot\frac{n-(q+4)}{2}\cdots\frac{n-(p+q+2)}{p}\times\frac{q+3}{1}\cdot\frac{q+2}{2}\cdots\frac{4}{q}\times\frac{n-(p+q+3)}{n-(p+q)}\cdot$$
$$\frac{n-(p+q+4)}{n-(p+q+1)}\cdots\frac{n-(p+q+r)}{n-(p+q+r-3)}\times\frac{3}{n-(p+q+r-2)}\cdot\frac{2}{n-(p+q+r-1)}$$
$$\frac{(n-(q+4)}{1}\cdot\frac{n-(q+5)}{2}\cdots\frac{n-(p+q+3)}{p}\times\frac{q+4}{1}\cdot\frac{q+3}{2}\cdots\frac{5}{q}\times\frac{n-(p+q+4)}{n-(p+q)}\cdot$$
$$\frac{n-(p+q+5)}{n-(p+q+1)}\cdots\frac{n-(p+q+r+1)}{n-(p+q+r-3)}\times\frac{4}{n-(p+q+r-2)}\cdot\frac{3}{n-(p+q+r-1)}$$
$$\vdots$$

$$\frac{p+r}{1}\cdot\frac{p+r-1}{2}\cdots\frac{r+1}{p}\times\frac{n-(p+r)}{1}\cdot\frac{n-(p+r+1)}{2}\cdots\frac{n-(p+q+r-1)}{q}\times\frac{r}{n-(p+q)}\cdot$$
$$\frac{r-1}{n-(p+q+1)}\cdots\frac{3}{n-(p+q+r-3)}\times\frac{n-(p+q+r)}{n-(p+q+r-2)}\cdot\frac{n-(p+q+r+1)}{n-(p+q+r-1)}$$
$$\frac{p+r-1}{1}\cdot\frac{p+r-2}{2}\cdots\frac{r}{p}\times\frac{n-(p+r-1)}{1}\cdot\frac{n-(p+r)}{2}\cdots\frac{n-(p+q+r-2)}{q}\times\frac{r-1}{n-(p+q)}$$
$$\cdot\frac{r-2}{n-(p+q+1)}\cdots\frac{2}{n-(p+q+r-3)}\times\frac{n-(p+q+r-1)}{n-(p+q+r-2)}\cdot\frac{n-(p+q+r)}{n-(p+q+r-1)}$$
$$\frac{p+r-2}{1}\cdot\frac{p+r-3}{2}\cdots\frac{r-1}{p}\times\frac{n-(p+r-2)}{1}\cdot\frac{n-(p+r-1)}{2}\cdots\frac{n-(p+q+r-3)}{q}\times\frac{r-2}{n-(p+q)}$$
$$\cdot\frac{r-3}{n-(p+q+1)}\cdots\frac{1}{n-(p+q+r-3)}\times\frac{n-(p+q+r-2)}{n-(p+q+r-2)}\cdot\frac{n-(p+q+r-1)}{n-(p+q+r-1)}.$$

$$=\frac{p+1.p+2\ldots p+r-2.q+1.q+2}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}$$
$$\times\frac{1.2\ldots p+q+1}{n-(p+q+r-1).n-(p+q+r-2)\ldots n+1}$$

$$\left\{\begin{array}{l}
\dfrac{n-(q+2).n-(q+3)\ldots n-(p+q+r-1)}{1.2\ldots p+r-2}\times\dfrac{1.2\ldots q+2}{1.2\ldots q+2}\\
\dfrac{n-(q+3).n-(q+4)\ldots n-(p+q+r)}{1.2\ldots p+r-2}\times\dfrac{2.3\ldots q+3}{1.2\ldots q+2}\\
\dfrac{n-(q+4).n-(q+5)\ldots n-(p+q+r+1)}{1.2\ldots p+r-2}\times\dfrac{3.4\ldots q+4}{1.2\ldots q+2}\\
\vdots\\
\dfrac{p+r}{1}\cdot\dfrac{p+r-1}{2}\cdots\dfrac{3}{p+r-2}\times\dfrac{n-(p+r).n-(p+r-1)\ldots n-(p+q+r+1)}{1.2\ldots q+2}\\
\dfrac{p+r-1}{1}\cdot\dfrac{p+r-2}{2}\cdots\dfrac{2}{p+r-2}\times\dfrac{n-(p+r-1).n-(p+r)\ldots n-(p+q+r)}{1.2\ldots q+2}\\
\dfrac{p+r-2}{1}\cdot\dfrac{p+r-1}{2}\cdots\dfrac{1}{p+r-2}\times\dfrac{n-(p+r-2).n-(p+r-1)\ldots n-(p+q+r-1)}{1.2\ldots q+2}
\end{array}\right.$$

Hence, the probability of the event (in a given order), is the sum of these last quantities; namely (§ 1)

$$\frac{p+1.p+2\ldots p+r-2.q+1.q+2}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}$$
$$\times\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$
$$\times\frac{n-(p+q+r-1).n-(p+q+r-2)\ldots n+1}{1.2\ldots p+q+r+1}$$
$$=\frac{p+1.p+2\ldots p+r-2.q+1.q+2}{p+q+2.p+q+3\ldots p+q+r+1}.$$

The probability to bring forth, in any order, $r-2$ white and 2 black, is

$$\cdots \frac{r}{1}\cdot\frac{r-1}{2}\times\frac{p+1.p+2\ldots p+r-2.q+1.q+2}{p+q+2.p+q+3\ldots p+q+r+1}.$$

And the probability to bring forth, in any order, 2 white & $r-2$ black, is

$$\cdots \frac{r}{1}\cdot\frac{r-1}{2}\times\frac{q+1.q+2\ldots q+r-2.p+1.p+2}{p+q+2.p+q+3\ldots p+q+r+1}.$$

4°. Seek the probability, that of r extracted tickets there will be of them $r-3$ white & three black.

The probabilities of this event corresponding to the first three & to the $r-3$ last hypotheses on the composition of the urn, vanish.

Namely, the number of white tickets, is contained between the numbers… $n-(q+3)$ & $p+r-3$.

And the number of black tickets, is contained between the numbers … $q+3\ldots n-(p+r-3)$.

The probabilities of the event (in a given order), corresponding to the remaining hypotheses, are respectively the products of the quantity

$$\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$

by the following quantities:

$$\frac{n-(q+3).n-(q+4)\ldots n-(p+q+2)}{1.2\ldots p}\times\frac{q+3.q+2\ldots 4}{1.2\ldots q}$$
$$\times\frac{n-(p+q+3).n-(p+q+4)\ldots n-(p+q+r-1)}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{3.2.1}{n-(p+q+r-3)\ldots n-(p+q+r-1)}$$
$$\frac{n-(q+4).n-(q+5)\ldots n-(p+q+3)}{1.2\ldots p}\times\frac{q+4.q+3\ldots 5}{1.2\ldots q}$$
$$\times\frac{n-(p+q+4).n-(p+q+5)\ldots n-(p+q+r)}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{4.3.2}{n-(p+q+r-3)\ldots n-(p+q+r-1)}$$
$$\frac{n-(q+5).n-(q+6)\ldots n-(p+q+4)}{1.2\ldots p}\times\frac{q+5.q+4\ldots 6}{1.2\ldots q}$$
$$\times\frac{n-(p+q+5).n-(p+q+6)\ldots n-(p+q+r+1)}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{5.4.3}{n-(p+q+r-3)\ldots n-(p+q+r-1)}$$
$$\vdots$$
$$\frac{p+r-1.p+r-2\ldots r}{1.2\ldots p}\times\frac{n-(p+r-1).n-(p+r)\ldots n-(p+q+r-2)}{1.2\ldots q}$$
$$\times\frac{r-1.r-2\ldots 3}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{n-(p+q+r-1)\ldots n-(p+q+r+1)}{n-(p+q+r-3)\ldots n-(p+q+r-1)}$$
$$\frac{p+r-2.p+r-3\ldots r-1}{1.2\ldots p}\times\frac{n-(p+r-2).n-(p+r-1)\ldots n-(p+q+r-3)}{1.2\ldots q}$$
$$\times\frac{r-2.r-3\ldots 2}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{n-(p+q+r-2)\ldots n-(p+q+r)}{n-(p+q+r-3)\ldots n-(p+q+r-1)}$$

$$\frac{p+r-3.p+r-4\ldots r-2}{1.2\ldots p}\times\frac{n-(p+r-3).n-(p+r-2)\ldots n-(p+q+r-4)}{1.2\ldots q}$$
$$\times\frac{r-3.r-4\ldots 1}{n-(p+q).n-(p+q+1)\ldots n-(p+q+r-4)}\times\frac{n-(p+q+r-3)\ldots n-(p+q+r-1)}{n-(p+q+r-3)\ldots n-(p+q+r-1)}.$$

$$=\frac{p+1.p+2\ldots p+r-3.q+1\ldots q+3}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}\times\frac{1.2\ldots p+q+1}{n-(p+q-1)\ldots n-(p+q-2)\ldots n+1}$$

$$\times\begin{cases}\dfrac{n-(q+3).n-(q+4)\ldots n-(p+q+r-1)}{1.2\ldots p+r-3}\times\dfrac{q+3.q+2\ldots 1}{1.2\ldots q+3}\\ \dfrac{n-(q+4).n-(q+5)\ldots n-(p+q+r)}{1.2\ldots p+r-3}\times\dfrac{q+4.q+3\ldots 2}{1.2\ldots q+3}\\ \dfrac{n-(q+5).n-(q+6)\ldots n-(p+q+r+1)}{1.2\ldots p+r-3}\times\dfrac{q+5.q+4\ldots 3}{1.2\ldots q+3}\\ \vdots\\ \dfrac{p+r-1.p+r-2\ldots 3}{1.2\ldots p+r-3}\times\dfrac{n-(p+r-1).n-(p+r)\ldots n-(p+q+r+1)}{1.2\ldots q+3}\\ \dfrac{p+r-2.p+r-3\ldots 2}{1.2\ldots p+r-3}\times\dfrac{n-(p+r-2).n-(p+r-1)\ldots n-(p+q+r)}{1.2\ldots q+3}\\ \dfrac{p+r-3.p+r-4\ldots 1}{1.2\ldots p+r-3}\times\dfrac{n-(p+r-3).n-(p+r-2)\ldots n-(p+q+r-1)}{1.2\ldots q+3}\end{cases}$$

Hence, the probability of the event (in a given order), is the sum of these last quantities; namely (§ 1)

$$\frac{p+1.p+2\ldots p+r-3\ldots q+1\ldots q+3}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}\times\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q-2)\ldots n+1}$$
$$\times\frac{n-(p+q-1).n-(p+q-2)\ldots n+1}{1.2\ldots p+q+r+1}=\frac{p+1.p+2\ldots p+r-3.q+1\ldots q+3}{p+q+2.p+q+3\ldots p+q+r+1}.$$

The probability to bring forth (in any order), $r-3$ white, & three black ...

$$\frac{r}{1}\cdot\frac{r-1}{2}\cdot\frac{r-2}{3}\times\frac{p+1.p+2\ldots p+r-3.q+1\ldots q+3}{p+q+2.p+q+3\ldots p+q+r+1}.$$

The probability to bring forth (in any order), 3 white & $r-3$ black ...

$$\frac{r}{1}\cdot\frac{r-1}{2}\cdot\frac{r-2}{3}\times\frac{q+1.q+2\ldots q+r-3.p+1\ldots p+3}{p+q+2.p+q+3\ldots p+q+r+1}.$$

Generally. Seek the probability that out of r tickets drawn, there will be, $r-m$ white & m black.

The probabilities of this event corresponding to the first m, & to the $r-m$ last hypotheses on the composition of the urn, vanish.

Namely, the number of the white tickets is contained between the numbers ... $n-(q+m)$ & $p+r-m$.

The number of black tickets is contained between the numbers $\ldots q+m$ & $n-(p+r-m)$.

The probabilities of this event (in a given order), corresponding to the remaining hypotheses, are respectively the products of the quantity

$$\frac{1.2\ldots p+q+1}{n-(p+q-1).n-(p+q+2)\ldots n+1}$$

by the following quantities:

$$\frac{n-(q+m).n-(q+m+1)\ldots n-(p+1+m-1)}{1.2\ldots p}\times\frac{q+m}{1}\cdot\frac{q+m-1}{2}\cdots\frac{m+1}{q}$$
$$\times\frac{n-(p+q+m)\ldots n-(p+q+r-1)}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{m.m-1\ldots 1}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$
$$\frac{n-(q+m+1).n-(q+m+2)\ldots n-(p+1+m)}{1.2\ldots p}\times\frac{q+m+1}{1}\cdot\frac{q+m}{2}\cdots\frac{m+2}{q}$$
$$\times\frac{n-(p+q+m+1)\ldots n-(p+q+r)}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{m+1.m\ldots 2}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$
$$\frac{n-(q+m+2).n-(q+m+3)\ldots n-(p+1+m+1)}{1.2\ldots p}\times\frac{q+m+2}{1}\cdot\frac{q+m+1}{2}\cdots\frac{m+3}{q}$$
$$\times\frac{n-(p+q+m+2)\ldots n-(p+q+r+1)}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{m+2.m+1\ldots 3}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$

$$\vdots$$

$$\frac{p+r-(m-2).p+r-(m-1)\ldots r-(m-3)}{1.2\ldots p}\times\frac{n-(p+r-(m-2)).n-(p+r-(m-3))\ldots n-(r-(m-3))}{1.2\ldots q}$$
$$\times\frac{r-(m-2)\ldots 3}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{n-(p+q+r-(m-2))\ldots n-(p+q+r+1)}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$
$$\frac{p+r-(m-1).p+r-m)\ldots r-(m-2)}{1.2\ldots p}\times\frac{n-(p+r-(m-1)).n-(p+r-(m-2))\ldots n-(r-(m-2))}{1.2\ldots q}$$
$$\times\frac{r-(m-1)\ldots 2}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{n-(p+q+r-(m-1))\ldots n-(p+q+r)}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$
$$\frac{p+r-m.p+r-(m+1)\ldots r-(m-1)}{1.2\ldots p}\times\frac{n-(p+r-m).n-(p+r-(m-1))\ldots n-(r-(m-1))}{1.2\ldots q}$$
$$\times\frac{r-m\ldots 1}{n-(p+q)\ldots n-(p+q+r-(m+1))}\times\frac{n-(p+q+r-m)\ldots n-(p+q+r-1)}{n-(p+q+r-m)\ldots n-(p+q+r-1)}$$

$$=\frac{p+1.p+2\ldots p+r-m.q+1\ldots q+m}{n-(p+q+r-1).n-(p+q+r-2)\ldots n-(p+q)}\times\frac{1.2\ldots p+q+1}{n-(p+q-1)\ldots n+1}$$

$$\times\begin{cases}\dfrac{n-(q+m).n-(q+m+1)\dots n-(p+q+r-1)}{1.2\dots p+r-m}\times\dfrac{q+m.q+m+1\dots 1}{1.2\dots q+m}\\ \dfrac{n-(q+m+1).n-(q+m+2)\dots n-(p+q+r)}{1.2\dots p+r-m}\times\dfrac{q+m+1.q+m\dots 2}{1.2\dots q+m}\\ \dfrac{n-(q+m+2).n-(q+m+3)\dots n-(p+q+r+1)}{1.2\dots p+r-m}\times\dfrac{q+m+2.q+m+1\dots 3}{1.2\dots q+m}\\ \vdots\\ \dfrac{p+r-(m-2).p+r-(m-1)\dots 3}{1.2\dots p+r-m}\times\dfrac{n-(p+r-(m-2)).n-(p+r-(m-3))\dots n-(p+q+r+1)}{1.2\dots q+m}\\ \dfrac{p+r-(m-1).p+r-m\dots 2}{1.2\dots p+r-m}\times\dfrac{n-(p+r-(m-1)).n-(p+r-(m-2))\dots n-(p+q+r)}{1.2\dots q+m}\\ \dfrac{p+r-m.p+r-(m+1)\dots 1}{1.2\dots p+r-m}\times\dfrac{n-(p+r-m).n-(p+r-(m-1))\dots n-(p+q+r-1)}{1.2\dots q+m}\end{cases}$$

Hence, the probability of the event (in a given order) is the sum of these last quantities; namely (§ 1)

$$\frac{p+1.p+2\dots p+r-m.q+1\dots q+m}{n-(p+q+r-1).n-(p+q+r-2)\dots n-(p+q)}\times\frac{1.2\dots p+q+1}{n-(p+q-1)\dots n+1}$$
$$\times\frac{n-(p+q+r-1).n-(p+q+r-2)\dots n+1}{1.2\dots p+q+r+1}$$
$$=\frac{p+1.p+2\dots p+r-m.q+1.q+2\dots q+m}{p+q+2.p+q+3\dots p+q+r+1}.$$

The probability to bring forth (in any order) $r-m$ white tickets & m black ...

$$\frac{r}{1}\cdot\frac{r-1}{2}\cdots\frac{r-(m-1)}{m}\times\frac{p+1\dots p+r-m.q+1\dots q+m}{p+q+2\dots p+q+r+1}$$

The probability to bring forth (in any order), m white tickets, & $r-m$ black...

$$\frac{r}{1}\cdot\frac{r-1}{2}\cdots\frac{r-(m-1)}{m}\times\frac{q+1\dots q+r-m.p+1\dots p+m}{p+q+2\dots p+q+r+1}.$$

§ 7. *Recapitulation.* One has drawn from an urn p white tickets, & q black tickets, by not replacing into the urn at each of the drawings the extracted ticket. One draws anew r tickets in the same manner. One obtains the following expressions of the probabilities that the numbers of white & black tickets will be as it follows.

Number of tickets white	black	Probabilities
r	0	$1\times\frac{p+1.p+2...p+r}{p+q+2.p+q+3...p+q+r+1}$
$r-1$	1	$\frac{r}{1}\times\frac{p+1.p+2...p+r-1.q+1}{p+q+2.p+q+3...p+q+r+1}$
$r-2$	2	$\frac{r.r-1}{1.2}\times\frac{p+1.p+2...p+r-2.q+1.q+2}{p+q+2.p+q+3...p+q+r+1}$
$r-3$	3	$\frac{r.r-1.r-2}{1.2.3}\times\frac{p+1.p+2...p+r-3.q+1...q+3}{p+q+2.p+q+3...p+q+r+1}$
$r-4$	4	$\frac{r.r-1...r-3}{1.2...4}\times\frac{p+1.p+2...p+r-4.q+1...q+4}{p+q+2.p+q+3...p+q+r+1}$
$\vdots$	$\vdots$	$\times$
$r-m$	m	$\frac{r.r-1...r-(m-1)}{1.2...m}\times\frac{p+1.p+2...p+r-m.q+1...q+m}{p+q+2.p+q+3...p+q+r+1}$
$\vdots$	$\vdots$	$\times$
4	$r-4$	$\frac{r.r-1...r-3}{1.2...4}\times\frac{p+1.p+2...p+4.q+1.q+2...q+r-4}{p+q+2.p+q+3...p+q+r+1}$
3	$r-3$	$\frac{r.r-1.r-2}{1.2.3}\times\frac{p+1.p+2.p+3.q+1.q+2...q+r-3}{p+q+2.p+q+3...p+q+r+1}$
2	$r-2$	$\frac{r.r-1}{1.2}\times\frac{p+1.p+2.q+1.q+2...q+r-2}{p+q+2.p+q+3...p+q+r+1}$
1	$r-1$	$\frac{r}{1}\times\frac{p+1.q+1.q+2...q+r-1}{p+q+2.p+q+3...p+q+r+1}$
0	r	$1\times\frac{q+1.q+2...q+r}{p+q+2.p+q+3...p+q+r+1}$

Remark. By § 2, the sum of these probabilities is equal to unity. Hence, the sum of the probabilities to bring forth r tickets in all the ways in which they can be combined as for two kinds of colors, is unity or certitude (thus as this must be).

§ 8. *Problem.* One demands the greatest of these probabilities. Let $r-m$, $r-(m+1)$ be two successive numbers of white balls to extract. The corresponding probabilities of their extraction are between them as the quantities

$$\frac{r.r-1.r-2\ldots r-(m-1)}{1.2.3\ldots m}\times p+1.p+2\ldots p+r-m\times q+1.q+2\ldots q+m$$

&

$$\frac{r.r-1.r-2\ldots r-m}{1.2.3\ldots m+1}\times p+1.p+2\ldots p+r-(m+1)\times q+1.q+2\ldots q+m+1;$$

or, as the quantities $p+r-m$ & $\frac{r-m}{m+1}(q+m+1)$.

But, when one of these probabilities is in the case of the *maximum*, this ratio is the one of each those of two successive probabilities which approach most to the ratio of equality; therefore, one of these probabilities being greatest, one approaches most to have the equation

$$p+r-m=\frac{r-m}{m+1}(q+m+1);$$

or the proportion

$$p+r-m:r-m=q+m+1:m+1;$$

from which one draws

$$p : q = r - m : m + 1$$
$$p : p + q = r - m : r + 1 . r - m = \frac{p}{p+1}(r+1).$$

If one has, $r - m = \frac{p}{p+q} r$; one will have $m = \frac{q}{p+q} r$. Hence, the ratio of $r - m$ to m approaches so much more to be the one of p to q, as r is greater relative to unity.

§ 9. The preceding formulas (§ 7) teach us that the probability of the future events, determined in kind, depend in no manner on the number of tickets contained in the urn, either before each extraction, or after some extractions; but that this probability depends uniquely on events already obtained.

This fact is so much more remarkable for the case where one does not return into the urn the extracted tickets, that is to say when the number of future events is diminished by the number of past events, that it no longer takes place when the total number of events remains constant; namely when one plays with one die of which the number of faces remains the same, or when one returns into the urn at each drawing the extracted tickets.

§ 10. The more the number of tickets contained originally in the urn is great, the smaller the influence that it has on the drawings following the extraction of a finite number of tickets not returned successively into the urn. Hence, the more the number of tickets contained originally in the urn is great, the more the probabilities to extract some tickets of given kinds, in given ratio, approaches to be the same; either that the number of the tickets in the urn remain constantly the same, or that this number is diminished successively by the tickets already extracted. Hence the preceding expressions apply themselves also to the case where the number of tickets remaining always the same, under the assumption that this number is very great. This conclusion agrees with that which I have developed immediately on these last probabilities, by supposing infinite the number of tickets, & of which I myself propose to make the subject of a following memoir. Under the assumption of the constancy of the tickets in the urn, the influence of the number of the tickets on the future events, although always real, is so much less as this number is greater.