SOLUTION
D'un problème de Probabilité*

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Having been charged, this year, with the course of Calculus of Probabilities which is given at the Faculté des Sciences, I have given, in this course, the solutions of many problems, among which I will cite the following, because of the curious results, and because I do not believe known, to what it leads.

Three players A, B, C, play, two by two, a sequences of trials; each new trial is played by the player who has won the preceding trial, with the one who has not played: lot designates the two players who play the first trial. The game is finished, when one of the three players has defeated consecutively the two others, or two trials in sequence; and it is this player who has won the game. One demands to determine, for the three players, the probabilities to win the game, according to the chances that they have to win at each trial, and according as the lot has designated them in order to play or in order to not play at the first trial.

When A and B play one against the other, I represent by γ the chance of A to win the trial and, consecutively, by $1-\gamma$ that of B. When these are C and A who play together, I represent likewise by β the chance of C and by $1-\beta$ that of A. Finally, when the trial is played between B and C, I designate by α the chance of B and by $1-\alpha$ that of C.

I suppose that the first trial is played between A and B, and won by A. It is easy to see that, n being any whole number, and as much that the game will not be finished, all the trials of which the rank is marked by a number of the form $3n-2$, will be played between A and B, and won by A; all those of which the rank is marked by a number of the form $3n-1$, will be played between C and A, and won by C; and all those of which the rank is marked by a number of the form $3n$, will be played between B and C, and won by B.

I call x_{3n-2} the probability that the game will not yet be terminated, at the trial of which the rank is $3n-2$, and x'_{3n-2} the probability that it will be terminated precisely at this trial. I designate likewise by y_{3n-1} the probability that the game will not be terminated in the first $3n-1$ trials, and by y'_{3n-1} the probability that it will end at the last of these trials. Finally, let z_{3n} be the probability that the game will not be terminated in the first $3n$ trials, and z'_{3n} the probability that it will end only at the trial of which the rank is 3n.

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These diverse notations, the ranks of the trials to which they correspond, the players who play at each of these trials, and their chances to win, will be easy things to be represented by this table:

$3n-2$,	$3n-1$,	$3n$,
x_{3n-2},	y_{3n-1},	z_{3n},
x'_{3n-2},	y'_{3n-1},	z'_{3n},
A and B,	C and A,	B and C,
γ and $1-\gamma$.	β and $1-\beta$.	α and $1-\alpha$.

The values of the quantities α, β, γ, are given, and are able to be extended from zero to unity. They will be all three equal to $\frac{1}{2}$, when the three players will be of equal force. They will be able to be all three greater than $\frac{1}{2}$: this case will take place, when A playing against B will be strongest; when C playing against A will also be strongest; when B playing against C will also be strongest, this which is not at all incompatible with the first two suppositions. We imagine, for example, that one has three urns A', B', C', containing some white balls and some black balls, and of which each contains more balls of the first color than of the second. We suppose also that the game between A and B, consists in drawing a ball from C', and that A wins, if this ball is white; that next, the game between C and A consists in drawing a ball from B', and that the chance favorable to C is the arrival of a white ball; and that finally, the game between B and C is to draw a ball from A', which will make B win, when it will be white: in this case, one will have

$$\gamma > \frac{1}{2}, \quad \beta > \frac{1}{2}, \quad \alpha > \frac{1}{2}.$$

It will be able to happen that unity or zero is the value of one of these quantities α, β, γ, of two of among them, or of all three: the case of $\gamma = 1$, $\beta = 1$, $\alpha = 1$, for example, will be the one where A will be certain to beat B, C to beat A, and B to beat C.

This put, in order that the game not be finished, at the trial of which the rank is $3n+1$ or $3(n+1)-2$, played between A and B, it is necessary, and it suffices 1°. that it not be at the preceding trial, played between B and C, and won by B; 2°. that the trial of which the rank is $3n+1$ is won by A. The probability x_{3n+1} of the game not terminated at this trial, will be therefore the product of the probability γ that it will be won by A, and of the probability z_{3n} that the game will not be finished at the trial of which the rank is $3n$, so that one will have

$$x_{3n+1} = \gamma z_{3n}.$$

One will find likewise

$$x'_{3n+1} = (1-\gamma) z_{3n};$$

because in order that the game end precisely at the trial of which the rank is $3n+1$, it will be necessary and it will be sufficient that it is not terminated at the preceding trial, and that the player B who will have won that one and of which $1-\gamma$ is the chance to beat A at the trial of which the rank is $3n+1$, wins effectively this second trial. By a similar reasoning, applied successively to the trials of which the ranks are $3n$ and $3n-1$, one will obtain these two other pairs of equations

$$z_{3n} = \alpha y_{3n-1}, \qquad z'_{3n} = (1-\alpha)y_{3n-1},$$
$$y_{3n-1} = \beta x_{3n-2}, \qquad y'_{3n-1} = (1-\beta)x_{3n-2},$$

which are deduced besides from the preceding pair, by simple changes of letters.

One draws from these equations

$$x_{3n+1}z_{3n}y_{3n-1} = \alpha\beta\gamma z_{3n}y_{3n-1}x_{3n-2};$$

and by making, for brevity,

$$\alpha\beta\gamma = k,$$

there results from it

$$x_{3n+1} = kx_{3n-2};$$

an equation in finite differences, of which the integral is found by putting successively $1, 2, 3, \ldots, n-1$, in the place of n, multiplying member by member the $n-1$ equations which will be obtained in this manner, and reducing; this which gives

$$x_{3n-2} = k^{n-1}x_1.$$

The factor x_1 is the arbitrary constant; at the first trial, or when $n = 1$, it is certain that the game is not terminated; one has therefore $x_{3n-2} = x_1 = 1$; and thence, and of the preceding equations, one concludes these values

$$x_{3n-2} = k^{n-1}, \qquad y_{3n-1} = \beta k^{n-1}, \qquad z_{3n} = \alpha\beta k^{n-1},$$
$$y_{3n-1} = (1-\beta)k^{n-1}, \qquad z'_{3n} = (1-\alpha)\beta k^{n-1}, \qquad x'_{3n+1} = (1-\gamma)\beta\alpha k^{n-1}.$$

These results will be verified easily in the extreme cases where one of the quantities α, β, γ will be zero or unity. If, for example, they are all three unity, the three probabilities $y'_{3n-1}, z'_{3n}, x'_{3n+1}$, will be null, and it will be certain that the game will never finish, this which is evident. If one of the fractions α, β, γ, is zero, one will have also $k = 0$; these three probabilities will be therefore null, except for $n = 1$; consequently, the game will be able to finish only at the second trial, or at the third, or at the fourth; the respective probabilities of these three events, will be

$$y'_2 = 1-\beta, \quad z'_3 = (1-\alpha)\beta, \quad x'_4 = (1-\gamma)\beta\alpha;$$

and as their sum is unity, because of $\alpha\beta\gamma = 0$, it follows that the game will certainly finish at one of these three trials.

In the case of $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, where the three players are of equal force, one has $k = \left(\frac{1}{2}\right)^3$. Whatever be the whole number m, the probability that the game will not end in the first m trials, becomes therefore $\left(\frac{1}{2}\right)^{m-1}$, according to the first three preceding equations; and the probability that it will end precisely at the m^{th} trial, becomes also $\left(\frac{1}{2}\right)^{m-1}$, according to the last three.

In the general case, where α, β, γ, are some fractions, if one represents by q_{3n-1}, the probability that the game will end at one of the n trials of which the ranks correspond to the numbers 2, 5, 8, 11, $\ldots 3n-1$; by r_{3n}, the probability that it will be terminated at one of the ranks corresponding to 3, 6, 9, 12, $\ldots 3n$; by p_{3n+1}, the probability that it will be terminated at one of the n trials of which the number 4, 7, 10, 12, $\ldots 3n+1$, mark the ranks, one will have

$$\begin{aligned} q_{3n-1} &= y'_2 + y'_5 + y'_8 + y'_{11} + \cdots + y'_{3n-1}, \\ r_{3n} &= z'_3 + z'_6 + z'_9 + z'_{12} + \cdots + z'_{3n}, \\ p_{3n+1} &= x'_4 + x'_7 + x'_{10} + x'_{13} + \cdots + x'_{3n+1}; \end{aligned}$$

and according to the preceding formulas, one will conclude from them

$$\begin{aligned} q_{3n-1} &= \frac{(1-\beta)(1-k^n)}{1-k}, \\ r_{3n} &= \frac{\beta(1-\alpha)(1-k^n)}{1-k}, \\ p_{3n+1} &= \frac{\alpha\beta(1-\gamma)(1-k^n)}{1-k}. \end{aligned}$$

If we call s_n the probability that the game will end in the first $3n$ trials, by departing from the second inclusively, we will have

$$s_n = q_{3n.-1} + r_{3n} + p_{3n+1};$$

and because of

$$1 - \beta + \beta(1-\alpha) + \alpha\beta(1-\gamma) = 1 - \alpha\beta\gamma = 1 - k,$$

there will result from it

$$s_n - 1 - k^n;$$

so that this probability will be the same, whatever be the two players who will play the first trial, and whatever be also, the player who will win it. It will not be likewise in regard to the probabilities that the game will end in the $3n-1$ or in the $3n+1$ first trials, not contained the first of all; probabilities which will be deduced from s_n, by subtracting from it the value of x'_{3n+1}, or by adding there that of y'_{3n+1} to it. If one excludes the case where the three quantities α, β, γ, are unity, and where one has $k = 1$, one sees that the probability that the game will not be prolonged beyond a number of given trials, will approach indefinitely unity in measure as this number will become greater, but that it would be changed into certitude only if this number became infinite. In the case of the players of equal force, by designating by m any whole number and by t_m the probability that the game will end in the first m trials by departing from the second inclusively, one will have

$$t_m = 1 - \left(\frac{1}{2}\right)^m;$$

such that it will suffice, for example, that one has $m = 10$, in order that the probability $1 - t_m$ of the contrary event, falls below a thousandth.

Now let a, b, c, be the probabilities that the game prolonged to infinity, if it is necessary it will be won respectively by A, B, C. In order that A win it is necessary and it suffices that the game is terminated at a trial at which the rank is marked by a number of the form $3n-1$; the value of a will be deduced therefore from that of q_{3n-1}, by making $n = \infty$; this which gives

$$a = \frac{1-\beta}{1-k},$$

by excluding the case where one has $k = 1$, and where the game never finishes. One will see likewise that b and c are the values of p_{3n+1} and r_{3n} which correspond to $n = \infty$; so that one has

$$b = \frac{\alpha\beta(1-\gamma)}{1-k}, \quad c = \frac{\beta(1-\alpha)}{1-k}$$

As it is certain that the game will be won by one of the three players, one must have

$$a + b + c = 1;$$

this which has place effectively.

Each of these fractions a, b, c, multiplied by the stake, or the sum of the three stakes, will be the mathematical expectation of one of the players; which will have advantage or disadvantage, according as this product will be greater or lesser than his stake. At any epoch whatsoever of the game, where the game is not finished, and where A just beat B; if the players agree to not complete, the stake must be apportioned among A, B, C, proportionally to the fractions a, b, c.

There results from their expressions, that even in the case of the players of equal force, the chance to win the game is unequal for the players who play first, and for the one who enters into the game only at the second trial. In fact, by making

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{1}{2}, \quad k = \frac{1}{8},$$

one will have

$$a = \frac{4}{7}, \quad b = \frac{1}{7}, \quad c = \frac{2}{7}.$$

After the first trial, the player who has won it has therefore right to $\frac{4}{7}$ of the stake, the one who has lost it has right only to $\frac{1}{7}$, and the one who has not played has right to $\frac{2}{7}$; but before this trial is played, the two players that the lot has designated in order to play it, have an equal chance of winning it; their probability to win the game is therefore then $\frac{1}{2}(a+b)$, or $\frac{5}{14}$, that is to say that it surpassed by $\frac{1}{14}$ the chance $\frac{2}{7}$, or $\frac{4}{14}$ of the player who enters only at the second trial. If the stake of each player is represented by μ, and if one agrees to not play the game, after the lot has designated the two players who should begin it, each of those here must take $\frac{15}{14}$ of μ, out of the stake equal to 3μ, and the third player $\frac{12}{14}$ of μ only; or alternately said, the three players having retired their stakes, the third must, beyond, give $\frac{1}{14}$ of μ to each of the first two.

In general, before the lot has designated the two players who must play the first trial, the probabilities to win the game will be different from a, b, c; for the three players A, B, C, I will represent them respectively by f, g, h; each of them will depend on the three given chances α, β, γ; and it will suffice to determine the value of f as function of α, β, γ: the values of g and h would be obtained in the same manner, or would be deduced from f by simple permutations.

It would be able to happen that the first trial is played by A and B, by C and A, by B and C. These three combinations being equally probable, the probability of each of them will be equal to $\frac{1}{3}$; moreover, in the first combination, the probability that the first trial will be won by A will have γ for value, and by B, it will be $1-\gamma$; in the second, the probability that C will win the first trial will be β, and the probability that it will be won by A will have $1-\beta$ for value; in the last, there will be the probability α that the first trial will be won by B, and the probability $1-\alpha$ that it will be by C: each of these three combinations giving place in two different cases, there will be therefore six possible cases, of which the respective probabilities will be

$$\frac{1}{3}\gamma, \quad \frac{1}{3}(1-\gamma), \quad \frac{1}{3}\beta, \quad \frac{1}{3}(1-\beta), \quad \frac{1}{3}\alpha, \quad \frac{1}{3}(1-\alpha).$$

Now, there will be question to determine successively, in each of the six cases, the probability that A will win the game; by multiplying next each probability by that of the case to which it corresponds, and making the sum of the six products, one will have the complete value of f.

In the first case, where the trial is played by A and B, and won by A, the probability that A will win the game will be the preceding value of a; the first term of the value of f will be therefore $\frac{1}{3}\gamma\alpha$, *or*

$$\frac{\gamma(1-\beta)}{3(1-k)}.$$

In the second case, where it is A playing against B, who loses the first trial, the probability that A will win the game will be the preceding value of b, in which one must change γ into $1-\gamma$, β into $1-\alpha$, α into $1-\beta$; by multiplying the result by $\frac{1}{3}(1-\gamma)$, one will have therefore

$$\frac{k'\gamma}{3(1-k')},$$

for the second term of f, where one has made, for brevity,

$$(1-\alpha)(1-\beta)(1-\gamma) = k'.$$

In the third case, where the first trial will be played by C and A, and won by C, the probability of A for winning the game will be that which the expression of b becomes, relative to the player who loses the first trial, when one changes γ into β, β into α, α into γ; by multiplying next by $\frac{1}{3}\beta$, there will result from it

$$\frac{k(1-\beta)}{3(1-k)},$$

for the third term of f.

In the fourth case, where the first trial is won by A playing against C, the probability that A will win the game will be the expression of a, in which it will be necessary to change γ into $1-\beta$, β into $1-\gamma$, α into $1-\alpha$: by multiplying the result by $\frac{1}{3}(1-\beta)$, one will have next

$$\frac{\gamma(1-\beta)}{3(1-k')},$$

for the fourth term of f.

In the fifth case, where the first trial is played by B and C, and won by B, the probability that A will win the game will be that which the expression of c becomes, relative to the player who does not play in this first trial, when one changes γ into α, β into γ, α into β; by multiplying the result by $\frac{1}{3}\alpha$, there comes

$$\frac{\alpha\gamma(1-\beta)}{3(1-k)},$$

for the fifth term of f.

Finally, in the sixth and last case, where the first trial is won by C playing against B, the probability that A will win the game will be deduced from the expression of c, but by changing γ into $1-\alpha$, β into $1-\beta$, α into $1-\gamma$; this which, after having multiplied by $\frac{1}{3}(1-\alpha)$, gives

$$\frac{\gamma(1-\alpha)(1-\beta)}{3(1-k')},$$

for the last term of f.

If one makes actually the sum of these six partial values of f, one will have, for its complete value,

$$f = \frac{\gamma(1-\beta)(1+\alpha+\alpha\beta)}{3(1-k)} + \frac{\gamma k' + \gamma(1-\beta) + \gamma(1-\alpha)(1-\beta)}{3(1-k')}.$$

The reason for the diverse permutations of the letters α, β, γ, that we just indicated, is easy to understand by casting the eyes on the table presented above. One will see likewise that in order to deduce from the expression of f, that of g, it will suffice to change γ into α, β into γ, α into β, in f; and in order to obtain next the value of h, it will be necessary to repeat again this permutation rotating into the value of g, or else to change immediately γ into β, β into α, α into γ, in the value of f In this manner, we will have

$$g = \frac{\alpha(1-\gamma)(1+\beta+\beta\gamma)}{3(1-k)} + \frac{\alpha k' + \alpha(1-\gamma) + \alpha(1-\beta)(1-\gamma)}{3(1-k')},$$

$$h = \frac{\beta(1-\alpha)(1+\gamma+\gamma\alpha)}{3(1-k)} + \frac{\beta k' + \beta(1-\alpha) + \beta(1-\gamma)(1-\alpha)}{3(1-k')}.$$

When the players are of equal force, or when one has $\alpha = \beta = \gamma = \frac{1}{2}$, these three probabilities f, g, h, must be equal among them and to $\frac{1}{3}$; this which one verifies easily. Whatever be the chances α, β, γ, it is necessary that one has

$$f + g + h = 1,$$

since it is certain that the game will be won by one of the three players, by excluding, however, the case where one will have $k = 1$, and where it would not end. It is also this which it is easy to verify, by having regard to that which k and k' represent.

If the two players B and C are of equal force, either when they play one against the other, or when they play against A, it will be necessary to make

$$\alpha = \frac{1}{2}, \quad \beta = 1 - \gamma;$$

the quantity γ will be yet able to be any fraction whatsoever; it will express the chance of A, to win each of the trials where he will play; and A will be stronger or more feeble than each of the two other players, according as one will have $\gamma > \frac{1}{2}$ or $\gamma < \frac{1}{2}$. For these values of α and β, one will have, as it must be

$$g = h = \frac{1}{2}(1 - f);$$

and the value of f will be reduced to

$$f = \frac{8\gamma^2 - 2\gamma^3}{3(2 - \gamma + \gamma^2)}.$$

If the stake is the same for each of the three players, and if one represents it by μ, the stake will be 3μ, and the mathematical expectation of A will have $3\mu f$ for value. It will be $2\mu\gamma$, if A always played with an equal stake, but in a single trial, and against one alone of the two other players; in the case of equal stakes, player A of unequal force, must therefore prefer to play in the game that we will consider, against the players B and C of equal forces, or else he must choose to play one simple game, against B or C, according as the difference $3\mu f - 2\mu\gamma$ is positive or negative. Now, according to the preceding value of f, one has

$$3\mu f - 2\mu\gamma = \frac{2\mu\gamma(2 - \gamma)(2\gamma - 1)}{2 - \gamma + \gamma^2};$$

a quantity positive or negative, according as γ surpasses $\frac{1}{2}$ or is less. It follows therefore that player A, if he is stronger, or if γ surpasses $\frac{1}{2}$, will increase yet his advantage, by choosing the first manner to play, and that if he is the weakest, or if one has $\gamma < \frac{1}{2}$, he will diminish his disadvantage, by choosing the second.

When the players, instead of putting, once for all, one sum into the game, agree to put one sum μ at each time that they enter, so that the stake increases continually with the number of trials, the mathematical expectation of each of them will be more the same as the preceding, and with equal force, for example, the advantage which was a little while ago for the players who play the first trial, will be now for the one who enters only at the second trial.

Finally to calculate conveniently the mathematical expectation of each player, it will be necessary to divide it into two parts: the one positive, and resulting from the sums that the player would be able to receive in the different trials which will be played; the other negative, and resulting from the sums that he would be able to pay. For player

A who wins the first trial, I will designate by a' the first part, by $a_\prime$ the second, setting aside the sign, and by ϕ the excess of the latter over the former. I will represent the analogous quantities by b', $b_\prime$, ψ, for player B who loses the first trial, and by c', $c_\prime, \theta$, for player C who enters only at the second trial. In this manner, one will have

$$\phi = a' - a_\prime, \quad \psi = b' - b_\prime, \quad \theta = c' - c_\prime.$$

At the m^{th} trial, if the game is not ended before, the stake will be equal to $(m+1)\mu$. Now, according to that which one has seen previously, the probabilities that the game will be finite, and won then by A, at the second, at the fifth, at the eleventh, etc., will be

$$(1-\beta), \quad (1-\beta)k, \quad (1-\beta)k^2, \quad (1-\beta)k^3, \quad \text{etc.;}$$

the gains attached to the arrived of these events being therefore

$$3\mu, \quad 6\mu, \quad 9\mu, \quad 12\mu, \quad \text{etc.,}$$

it follows from the rule of mathematical expectation, that the complete value of a' will be the sum of these two series multiplied term by term and extended to infinity; this which gives

$$a' = 3\mu(1-\beta)\sum ik^{i-1};$$

the sum $\sum$ being extended to all the values of the whole number i, from then $i = 1$ to $i = \infty$. But one has

$$\sum k^i = \frac{k}{1-k};$$

and by differentiating with respect to k, there comes

$$\sum ik^{i-1} = \frac{1}{(1-k)^2};$$

consequently, one will have

$$a' = \frac{3\mu(1-\beta)}{(1-k)^2}.$$

For A and B, there is certitude to play at the first trial, and for C, at the second. For A, the probabilities to reenter into the game, at the fourth trial, at the seventh, at the tenth, at the thirteenth, etc., or, this which is the same thing, the probabilities that the game will not be terminated at the third trial, at the sixth, at the ninth, at the twelfth, etc., will be, as one has found above,

$$\alpha\beta, \quad \alpha\beta k, \quad \alpha\beta k^2, \quad \alpha\beta k^3, \quad \text{etc.;}$$

the complete value of $a_\prime$ will be therefore the sum of this infinite series of fractions, multiplied by μ, and increased by μ for the first stake of A; so that one will have

$$a_\prime = \mu + \frac{\mu\alpha\beta}{1-k}.$$

The probability that the game will be terminated at the trial of which the rank is marked by a number of the form $3n+1$, in which case it will be won by B, being

$$\alpha\beta(1-\gamma)k^{n-1},$$

and the stake that B will then receive, having $3n\mu+2\mu$ for value, one concludes from it that the complete value of b', will be

$$b' = \mu\alpha\beta(1-\gamma)\left(3\sum nk^{n-1} + 2\sum k^{n-2}\right);$$

the sums $\sum$ being extended from $n=1$ to $n=\infty$. Therefore, by having regard to the values of these two sums, we will have

$$b' = \frac{3\mu\alpha\beta(1-\gamma)}{(1-k)^2} + \frac{2\mu\alpha\beta(1-\gamma)}{1-k}.$$

By a similar reasoning, one will find likewise

$$c' = \frac{3\mu\beta(1-\alpha)}{(1-k)^2} + \frac{\mu\beta(1-\alpha)}{1-k}.$$

One will obtain also, without difficulty,

$$b_{\prime} = \mu + \frac{\mu\beta}{1-k}, \quad c_{\prime} = \mu + \frac{\mu\alpha\beta\gamma}{1-k};$$

and from these diverse values, there will result finally

$$\begin{aligned}
\phi &= \frac{3\mu(1-\beta)}{(1-k)^2}\text{-}\mu + \frac{\mu\alpha\beta}{1-k},\\
\psi &= \frac{3\mu\alpha\beta(1-\gamma)}{(1-k)^2} + \frac{2\mu\alpha\beta(1-\gamma)}{1-k} - \mu - \frac{\mu\beta}{1-k},\\
\theta &= \frac{3\mu\beta(1-\alpha)}{(1-k)^2} + \frac{\mu\beta(1-\alpha)}{1-k} - \mu - \frac{\mu\alpha\beta\gamma}{1-k}.
\end{aligned}$$

Since all the money put successively into the game during the duration of the game, is retired by the player who has won it, it follows that the sum of the mathematical expectations of the three players must be null; and in effect, according to that which k represents, one has identically

$$\phi + \psi + \theta = 0.$$

By putting $\frac{1}{2}$ for each of the fractions α, β, γ, ad by making $k = \frac{1}{8}$, one has

$$\phi = \frac{33\mu}{49}, \quad \psi = -\frac{39\mu}{49}, \quad \theta = \frac{6\mu}{49},$$

for the mathematical expectations of the three players supposed of equal force. That signifies, for example, that before they had nothing put into the game, and after the first trial has been played; if one agrees to not continue the game, the player who has lost this first trial must pay $\frac{39}{49}$ of μ, namely, $\frac{33}{49}$ to the one who has won it, and $\frac{6}{49}$ to the one who has not played. After the lot has designated the two players who must play the first trial, and before this trial is played, each of them is able equally to win; the mathematical expectation of each of these players is therefore $\frac{1}{2}\cdot\frac{33\mu}{49}-\frac{1}{2}\cdot\frac{39\mu}{49}$, or $-\frac{3\mu}{49}$, that is to say, that if one agrees to not play the game, each of them must give $\frac{3}{49}$ of μ to the third player, while in the case that we have first examined, it was this first player who must, on the contrary, give $\frac{1}{14}$ of μ to each of the first two players. If the game is not terminated at the m^{th} trial, and if one agrees to not continue it, the player who will have lost this trial must, as we just said of it, pay $\frac{33}{49}$ of μ to the one who will win it, and $\frac{6}{49}$ to the third player; and moreover, the stake which had place at the preceding trial, and which was raised to μu, must be divided among these three players, proportionally to their chances $\frac{1}{7}$, $\frac{4}{7}$, $\frac{2}{7}$ of achieving to win the game; so that at the m^{th} trial, if one represents by ϕ' the mathematical expectation of the player who wins this trial, by ψ' that of the player who loses it, by θ' that of the player who does not play, one will have

$$\phi' = \frac{(28m+33)\mu}{49}, \quad \psi' = \frac{(7m-39)\mu}{49}, \quad \theta' = \frac{(14m+6)\mu}{49}.$$

When m will surpass five, the value of ψ' will be positive, as ϕ' and θ'; the player who loses the m^{th} trial, will have nothing to pay to the two others; only he will have right to a smaller part in the stake which had place in the preceding trial: if, for example, one has $m=6$, the stake which had place at the fifth trial will be equal to 6μ; out of what the player who wins the sixth trial must take $\frac{201\mu}{49}$, the one who loses it will have right only to $\frac{3\mu}{49}$, and the third player to $\frac{90\mu}{49}$.