Suite du Mémoire sur la probabilité du résultat moyen des observations, inséré dans la Connaissance des Tems de l'année 1827.[*]

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I myself propose to add some new developments to the part of this Memoire which is relative to the probability of the arithmetic mean among the results of a great number of observations.

1. We suppose that any one thing, that we will name A in order to shorten the discourse, is susceptible, by its nature, to all the values contained between some given limits and represented by a and b. Let x be one of these values; if one makes a series of experiences in order to determine A, the probability that the value that one will find through one of these observations will not exceed x, will vary, in general, from one experience to the other. We will represent it by $F_n x$ for the nth experience. The probability that this value will be precisely x will be able to be only infinitely small, since the number of possible values is infinite; by making $\frac{F_n x}{dx} = f_n x$, it will have for expression $f_n x dx$.

We designate by X a given function of x, which increases without interruption from $x = a$ to $x = b$, and we represent by a_1 and b_1 its extreme values. For greater generality, we are going to seek the probability that if one takes the sum of the values of X which will result from a number s of successive observations, this sum will be contained between some given limits.

We admit first that X is only susceptible of a number ν of equi-different values; we will make next ν infinite, and the difference of two consecutive values infinitely small. We suppose then that a_1 and b_1 are some multiples of one same quantity w, so that $a_1 = p_1 w$, $b_1 = q_1 w$, p_1 and q_1 being some whole numbers, positive or negative. We designate by iw one of the intermediate values of X, i being also a whole number or zero; by making $q_1 - p_1 = \nu - 1$, the number of values of X will be equal to ν, and their constant difference equal to w. We call Q_n the probability of the value of x which corresponds to $X = iw$, relative to the nth observation. Finally let M be the probability that in a number s of observations, the sum of the values of X will be equal to mw, m being a whole number contained between sp_1 and sq_1. It is easy to see that M will be

the coefficient of t^m in the development of

$$\sum t^i Q_1 . \sum t^i Q_2 . \sum t^i Q_3 \dots \sum t^i Q_s,$$

according to the powers of t; each of the sums $\sum$ extending themselves to all the values of i contained from p_1 to q_1, and being composed, consequently, of a number ν of terms. One is able to say that M will be the part independent of t in the product of this function of t multiplied by t^{-m}; and if one puts in this product $e^{\theta\sqrt{-1}}$ in place of t, if one makes, for brevity,

$$\sum e^{i\theta\sqrt{-1}} Q_1 . \sum e^{i\theta\sqrt{-1}} Q_2 . \sum e^{i\theta\sqrt{-1}} Q_3 \dots \sum e^{i\theta\sqrt{-1}} Q_s = P,$$

one will conclude from it

$$M = \frac{1}{2\pi} \int_{-\pi}^{\pi} P e^{-m\theta\sqrt{-1}} d\theta;$$

e being the base of the Naperian logarithms, and π the ratio of the circumference to the diameter.

We represent by p the probability that the same sum of ν values of X will be contained between μw and $\mu' w$, μ and μ' being some whole numbers or zero, which will not exit from some limits sp_1 and sq_1. It is evident that p will be the sum of the values of M that one will obtain by giving successively to m all the values contained from $m = \mu$ to $m = \mu'$ inclusively. Now, by having regard to the sum of the values corresponding to the factor $e^{-m\theta\sqrt{-1}}$, there comes

$$p = \frac{1}{4\pi\sqrt{-1}} \int_{-\pi}^{\pi} \left[e^{-(\mu-\frac{1}{2})\theta\sqrt{-1}} - e^{-(\mu'-\frac{1}{2})\theta\sqrt{-1}} \right] \frac{d\theta}{\sin\frac{1}{2}\theta}.$$

Let finally,

$$\mu w = c - \varepsilon, \quad \mu' w = c + \varepsilon, \quad \frac{\theta}{w} = \alpha;$$

there will result from it

$$= \frac{w}{2\pi} \int P \frac{\sin(\varepsilon + \frac{1}{2}w)\alpha}{\sin(\frac{1}{2}w\alpha)} e^{-c\alpha\sqrt{-1}} d\alpha;$$

and the limits relative to α will be $\pm\frac{\pi}{w}$. In the case of ν infinity, or of w infinitely small, they will become $\pm\infty$; one will be able to replace $\varepsilon+\frac{1}{2}w$ by ε, and $\frac{2}{w\alpha}\sin\frac{1}{2}w\alpha$ by unity; by means of which the expression of p will be changed into this here:

$$p = \frac{1}{\pi} \int_{-\infty}^{\infty} P e^{-c\alpha\sqrt{-1}} \sin\varepsilon\alpha \frac{d\alpha}{\alpha} \qquad (1)$$

At the same time the quantities iw and Q_n will coincide with X and $f_n x\,dx$; the sums $\sum$ contained in P will be transformed into some definite integrals relative to x, of which the limits will be a and b; and one will have

$$P = \int_a^b f_1 e^{-X\alpha\sqrt{-1}} x\,dx . \int_a^b f_2 e^{-X\alpha\sqrt{-1}} x\,dx \dots \int_a^b f_s e^{-X\alpha\sqrt{-1}} x\,dx. \qquad (2)$$

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2. This formula (1) will express, in the most general manner, the probability that the sum of s values of the function X, resulting from a parallel number of successive observations, will be contained between $c - \varepsilon$ and $c + \varepsilon$, which are some quantities given and contained between sa_1 and sb_1. If one makes there $X = x$, p will be the probability that the value of A, expressed by the mean result of these s observations, will have for limits $\frac{1}{s}(c \pm \varepsilon)$. As the result of each observation must be contained, by hypothesis, between a and b, it will be necessary that one has

$$\int_a^b f_1 x \, dx = 1, \quad \int_a^b f_2 x \, dx = 1, \ldots \int_a^b f_s x \, dx = 1. \tag{3}$$

The quantities $f_1 x$, $f_2 x$, etc., will be besides some functions any whatever of x, provided that their values be all positive or not surpass unity. When these functions will be given, one will be able to calculate the exact value of p; but most often the law of probability of the values of A is unknown and variable from one observation to the other: the s functions $f_1 x$, $f_2 x$, etc., are therefore so many unknowns; this which does not prevent nevertheless that one is able, in the case where the number of observations is considerable, to deduce from the preceding formulas a value of p so much closer as the number s will be great.

When one has $c - \varepsilon = sa_1$ and $c + \varepsilon = sb_1$ the limits to which the probability p corresponds are the same limits a_1 and b_1 which comprehend, by hypothesis, the unknown value of X; p must therefore be certitude, or equal to unity; and it is, in fact, that which one is able to verify.

For this, I replace X and x by X_1 and x_1, X_2 and $x_2 \ldots X_s$ and x_s, in the first, the second, ... the last of the s integrals of which p is the product; I substitute next its expression into that of p; and by making

$$X_1 + X_2 + X_3 \ldots + X_s = \sigma,$$

equation (1) becomes

$$p = \frac{1}{\pi} \int_a^b \int_a^b \cdots \int_a^b \int_{-\infty}^{\infty} \left(e^{(\sigma - c)\alpha\sqrt{-1}} \sin \varepsilon \alpha \, \frac{d\alpha}{\alpha} \right) f_1 x_1 f_2 x_2 \ldots f_s x_s \, dx_1 dx_2 \ldots dx_s$$

Now, we have

$$\int_{-\infty}^{\infty} e^{(\sigma - c)\alpha\sqrt{-1}} \sin \varepsilon \alpha \, \frac{d\alpha}{\alpha} = \frac{1}{2} \int_{-\infty}^{\infty} \sin(\varepsilon + \sigma - c)\alpha \, \frac{d\alpha}{\alpha}$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} \sin(\varepsilon - \sigma + c)\alpha \, \frac{d\alpha}{\alpha}$$

According to the limits of the integrals relative to $x_1, x_2, \ldots x_s$, the sum σ is not able to be less than sa_1, nor surpass sb_1; in the case that we examine, the two coefficients $\varepsilon + \sigma - c$ and $\varepsilon - \sigma + c$ are therefore positive: consequently, these last two integrals are, as one knows, one and the other equal to π: one has therefore

$$\int_{-\infty}^{\infty} e^{(\sigma - c)\alpha\sqrt{-1}} \sin \varepsilon \alpha \, \frac{d\alpha}{\alpha} = \pi;$$

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whence there results

$$p = \int_a^b \int_a^b \cdots \int_a^b f_1 x_1 f_2 x_2 \ldots f_s x_s \, dx_1 dx_2 \ldots dx_s;$$

a quantity which is reduced to unity, by virtue of equations (3).

3. From this that the integral $\int_a^b f_n x dx$ is unity, and that $f_n x$ has only positive values, it follows that the integrals $\int_a^b f_n x \cos \alpha X \, dx$ and $\int_a^b f_n x \sin \alpha X \, dx$ are less than unity; so that one is able to put

$$\left. \begin{array}{l} \displaystyle\int_a^b f_n x \, \cos \alpha X \, dx = \rho_n \cos \phi_n, \\[2mm] \displaystyle\int_a^b f_n x \sin \alpha X \, dx = \rho_n \sin \phi_n, \end{array} \right\} \tag{4}$$

ρ_n and ϕ_n being some real quantities, of which the first will be regarded as positive. By making next

$$\rho_1 \rho_2 \rho_3 \cdots \rho_s = R,$$
$$\phi_1 + \phi_2 + \phi_3 \cdots + \phi_s = \psi,$$

formula (2) will become

$$P = R e^{\psi \sqrt{-1}}$$

For two values of α equal and of contrary sign, there will be likewise in regard to the corresponding values of the angle ψ, and those of the quantity R will be equal and of the same sign. After this consideration, and by means of the value of P, formula (1) will be changed into this here:

$$P = \frac{2}{\pi} \int_0^\infty R \cos(\psi - c\alpha) \sin \varepsilon \alpha \frac{d\alpha}{\alpha}. \tag{5}$$

Each of the factors of R is equal to unity for $\alpha = 0$, and < 1 for every other value of α. In fact, the expression of the square of ρ_n is able to be written thus:

$$\rho_n^2 = \int_a^b f_n x \, \cos \alpha X \, dx. \int_a^b f_n x' \, \cos \alpha X' \, dx'$$
$$+ \int_a^b f_n x \sin \alpha X \, dx. \int_a^b f_n x' \sin \alpha X' \, dx',$$

by designating by X' that which X becomes when one puts there x' in the place of x; now, this equation is the same thing as

$$\rho_n^2 = \int_a^b \int_a^b f_n x f_n x' \, \cos \alpha (X - X') \, dx dx';$$

and it is evident that the value of ρ_n will be less than the square root of $\int_a^b \int_a^b f_n x f_n x' \, dx dx'$, or than $\int_a^b f_n x \, dx$, and consequently less than unity. There results from this that when the number s of its factors will be very great, the product R will have sensible

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values only for some very small values of α. It is for this reason that one is able then to obtain a value near to the integral relative to α, that contains formula (5).

4. If we make, for brevity,

$$\int_a^b X f_n x \, dx = k_n, \quad \int_a^b X^2 f_n x \, dx = k'_n, \quad \text{etc.,}$$

and if we develop the first members of equations (4) according to the powers of α, we will have

$$\rho_n \cos \phi_n = 1 - \frac{\alpha^2}{2} k'_n + \frac{\alpha^4}{2.3.4} k'''_n - \text{etc.}$$

$$\rho_n \sin \phi_n = \alpha k_n - \frac{\alpha^3}{2.3} k''_n + \text{etc.}$$

The quantities k_n, k'_n, k''_n, etc., will increase less rapidly than the powers $(b_1 - a_1)$, $(b_1 - a_1)^2$, $(b_1 - a_1)^3$, etc.; this which suffices in order that these developments are of some series which will always end by being convergent, and consequently in order that one be able to employ them in place of $\rho_n \cos \phi_n$ and $\rho_n \sin \phi_n$. One deduces from them for ρ_n and ϕ_n of some series, of which the one contains only some even powers and the other odd powers of α, namely:

$$\rho_n = 1 - \alpha^2 h_n + \alpha^4 l_n - \text{etc.,}$$

$$\phi_n = \alpha k_n - \alpha^3 g_n + \text{etc.,}$$

by making, for brevity,

$$\frac{1}{2}(k'_n - k_n^2) = h_n,$$

$$\frac{1}{6}(k''_n - 3k_n k'_n + 2k_n^3) = g_n,$$

etc.,

and one concluded thence

$$\log \rho_n = -\alpha^2 h_n + \alpha^4 \left(l_n - \frac{1}{2} h_n^2 \right) + \text{etc.,}$$

$$\rho_n = e^{-\alpha^2 h_n} \left[1 + \alpha^4 \left(l_n - \frac{1}{2} h_n^2 \right) + \text{etc.,} \right]$$

Let further, for brevity,

$$\sum k_n = ks, \quad \sum h_n = hs, \quad \sum \left(l_n - \frac{1}{2} h_n^2 \right) = ls, \quad \text{etc.;}$$

the sums $\sum$ being extended from $n = 1$ to $n = s$. There will result from it

$$K = e^{-\alpha^2 hs} (1 + \alpha^4 ls + \text{etc.}),$$

$$\psi = \alpha ks - \alpha^3 gs + \text{etc.,}$$

$$\cos(\psi - c\alpha) = \cos(ks - c)\alpha + \alpha^3 gs \, \sin(ks - c)\alpha + \text{etc.}$$

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The quantities k, h, g, etc., will be able to vary with s; but they will not increase indefinitely with this number, and they will always form, as the integrals k_u, k_u', k_u'', etc., from which they are deduced, a series less increasing than that of the powers of $b_1 - a_1$.

I substitute these values into formula (5); if make besides

$$\alpha = \frac{\beta}{\sqrt{s}}, \quad d\alpha = \frac{d\beta}{\sqrt{s}};$$

and I neglect the terms of this formula which will be of the order of smallness of $\frac{1}{s}$, that is to say the terms which will be divided by s outside of sinus and cosinus: there comes

$$
\left.
\begin{aligned}
p = {} & \frac{2}{\pi} \int_0^\infty e^{-\beta^2 h} \cos \frac{(ks-c)\beta}{\sqrt{s}} \sin \frac{\varepsilon\beta}{\sqrt{s}} \frac{d\beta}{\beta} \\
& + \frac{2g}{\pi\sqrt{s}} \int_0^\infty e^{-\beta^2 h} \sin \frac{(ks-c)\beta}{\sqrt{s}} \sin \frac{\varepsilon\beta}{\sqrt{s}} \beta^2 d\beta
\end{aligned}
\right\}
\tag{6}
$$

In order that these integrals not be indeterminates, it is necessary that h be a positive quantity; and it is also that which takes place. In effect, according to that which k_n and k_n' represent, one has

$$2h_n = \int_a^b X^2 f_n x\, dx \int_a^b f_n x'\, dx' - \int_a^b X f_n x\, dx \int_a^b X' f_n x'\, dx';$$

a quantity which one is able to reduce to a single double integral, namely:

$$2h_n = \int_a^b \int_a^b (X^2 - XX') f_n x f_n x'\, dx dx',$$

or, this which is the same thing,

$$2h_n = \int_a^b \int_a^b (X'^2 - XX') f_n x f_n x'\, dx dx'.$$

Now, by adding these two equations, one has

$$4h_n = \int_a^b \int_a^b (X - X')^2 f_n x f_n x'\, dx dx';$$

and this value of $4h_n$ is evidently positive, and is not able to be null, since all the elements of the double integral are positive. Therefore, it will be likewise in regard to it of $\sum h_n$ and of h. That being, one will obtain by the known rules the exact value of the second integral contained in formula (6), and one will reduce, if one wishes, the first to a simpler form.

5. If one takes $c = \varepsilon$, p will be the probability that the sum of the values of X given by the s observations, will not exit the limits zero and 2ε. By differentiating with respect to ε, one will have

$$
\begin{aligned}
\frac{dp}{ds} = {} & \frac{2}{\pi\sqrt{s}} \int_0^\infty e^{-\beta^2 h} \cos \frac{(2\varepsilon - ks)\beta}{\sqrt{s}} d\beta \\
& - \frac{2g}{\pi\sqrt{s}} \int_0^\infty e^{-\beta^2 h} \sin \frac{(2\varepsilon - ks))\beta}{\sqrt{s}} \beta^3 d\beta;
\end{aligned}
$$

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and $\frac{dp}{ds}ds$ will be the infinitely small probability that the sum of the values of X will be precisely equal to 2ε.

We make now
$$2\varepsilon = ks + 2u\sqrt{hs};$$

we will have
$$\int_0^\infty e^{-\beta^2 h}\cos(2u\beta\sqrt{h})d\beta = \frac{\sqrt{\pi}}{2\sqrt{h}}e^{-u^2};$$

whence one draws, by differentiating with respect to u,
$$\int_0^\infty e^{-\beta^2 h}\sin(2\beta u\sqrt{h})\beta^3 d\beta = \frac{\sqrt{\pi}}{4h^2}(3y - 2u^3)e^{-u^2}.$$

Because $\frac{dp}{du} = \frac{dp}{d\varepsilon}\sqrt{hs}$, one will have therefore
$$\frac{dp}{du} = \frac{1}{\sqrt{\pi}}e^{-u^2} - \frac{g}{4h\sqrt{hs}}(3u - 2u^3)e^{-u^2}; \qquad (7)$$

and if one designates by X_n the value of X which will be given by the nth observation, $\frac{dp}{du}du$ will be the probability that one will have
$$\sum X_n = ks + 2u\sqrt{hs}; \qquad (8)$$

the sum $\sum$ being extended to all the observations. I integrate $\frac{dp}{du}du$ between some given limits, which I will represent by $\pm\gamma$; there comes
$$p = \frac{2}{\sqrt{u}}\int_0^\gamma e^{-u^2}du, \qquad (9)$$

for the probability that $\sum X_n$ will be contained between the limits $ks \pm 2\gamma\sqrt{hs}$, and the mean value of X, or $\frac{1}{s}\sum X_n$, between those here:
$$k \pm \frac{2\gamma\sqrt{h}}{\sqrt{s}}.$$

It is also this which one will deduce from equation (6), by making there
$$c = ks, \quad \varepsilon = 2\gamma\sqrt{hs},$$

and effecting the integrations.

One will always be able to take γ great enough in order that the value of p differs as little as one will wish from unity. For $\gamma = 3$, for example, one will have
$$\int_\gamma^\infty e^{-u^2}du = 0.000019577,$$

according to the table of values of this integral, which is found at the end of the *Analyse des réfractions* de Kramp; and as one has
$$\int_0^\gamma e^{-u^2}du = \frac{1}{2}\sqrt{\pi} - \int_\gamma^\infty e^{-u^2}du,$$

there will result from it

$$p = 1 - 0.000022091;$$

this which differs very little from certitude. One is able therefore to regard as extremely probable that the value of $\frac{1}{s}\sum X_n$ resulting from the observations, will approach indefinitely to be equal to k, and that by taking it for the value of k, the error to fear will be less than $\frac{2\gamma\sqrt{h}}{\sqrt{s}}$ on each side, γ being a number of little importance.

It is good to observe that the terms divided by s which have been neglected in passing from equation (5) to formula (6), would have for factor $e^{-\gamma^2}$ after the integrations relative to u; this which contributes again to render them very small; independently of the magnitude of s; because for $\gamma = \frac{3}{2}$, for example, the factor $e^{-\gamma^2}$ is below two thousandths, and it diminishes very rapidly for greater values of γ.

6. The curve of which the equation is

$$y = f_n x,$$

represents the law of probability of the values of A in the nth observation, in this sense that the element ydx of its area is the probability of the value of A expressed by the abscissa corresponding to x, and the same area, the probability that this value will not be $> x$. That which has for equation

$$y = \frac{1}{s}\sum f_n x,$$

is the curve of mean probability, relative to the series of s observations. According to equations (3), its total area, from $x = a$ to $x = b$, will be equal to unity; and by calling x_1, the abscissa of its center of gravity, one will have

$$\frac{1}{s}\sum \int_a^b x f_n x dx = x_1.$$

Now, if one makes $X = x$ in the expression of k_n of n° 4, there results from it

$$k_n = \int_a^b x f_n x dx, \quad k = \frac{1}{s}\sum \int_a^b x f_n x dx = x_1$$

this abscissa x_1 is therefore, in every case, the limit of which the mean result of a series of observations approaches indefinitely. In designating by λ_n the particular value of A which will be given by the nth observation, the mean result of which there is question will be $\frac{1}{s}\sum \lambda_n$; there will be there the probability p, given by formula (9), that its value will be contained between the limits

$$x_1 \pm \frac{2\gamma\sqrt{h}}{\sqrt{s}};$$

and if one makes also $X = x$ in the expression of h of n° 4, one will have

$$h = \frac{1}{2s}\sum \left[\int_a^b x^2 f_n x dx - \left(\int_a^b x f_n x dx \right)^2 \right]. \tag{10}$$

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One is able to present this result under another form, by making, in equation (9)

$$u\sqrt{h} = v, \quad \gamma\sqrt{h} = \delta;$$

this which gives

$$p = \frac{2}{\sqrt{\pi h}} \int_0^\delta e^{-\frac{v^2}{h}} dv, \tag{11}$$

for the probability that the value of $\frac{1}{s}\sum\lambda_n$ will be contained between the limits

$$x_1 \pm \frac{2\delta}{\sqrt{s}}.$$

The infinitely small probability of an intermediate value $x_1 + \frac{2v}{\sqrt{s}}$, would be deduced from formula (7), by putting there $\frac{v}{\sqrt{h}}$ in the place of v, and by multiplying by $\frac{dv}{\sqrt{h}}$. One sees that for a given value of δ, it would depend on two unknown quantities h and g, while the probability of the preceding limits, which it will suffice us to know, will depend only on a sole unknown h, of which it remains to us to calculate the value according to the results given from s observations.

7. For this, let

$$x = x_1 + z, \quad f_n x = f'_n z, \quad a = x_1 + a', \quad b = x_1 + b';$$

we will have

$$\int_{a'}^{b'} f'_n z\, dz = 1, \quad \int_{a'}^{b'} z f'_n z\, dz = 0;$$

equation (10) will become

$$h = \frac{1}{2s}\sum \int_{a'}^{b'} z^2 f'_n z\, dz;$$

and if we take

$$X = (x - x_1)^2 = z^2,$$

the quantity k of n° 4 will be the double of that value of h.

According to formula (7), the infinitely small probability of equation (8) is of the form:

$$\frac{du}{\sqrt{\pi}} e^{-u^2} + uU\, du,$$

U being a function of u, equal and of the same sign for two values of u equal and of contrary sign, of which the value is of the order of $\frac{1}{\sqrt{s}}$. By applying this equation (8) to the preceding value of X, and putting there, consequently, $2h$ in the place of k, one will deduce from it

$$h = \frac{1}{2s}\sum(\lambda_n - x_1)^2 + u\sigma;$$

σ being a quantity independent of u, which will also be of the order of $\frac{1}{\sqrt{s}}$. The same formulas (7) and (8), applied to the case of $X = x$, give

$$x_1 = \frac{1}{s}\sum\lambda_n + u'\sigma',$$

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and, for the probability of this equation,

$$\frac{du'}{\sqrt{\pi}}e^{-u'^2} + u'U'du';$$

σ' and U' being some quantities of the order of $\frac{1}{\sqrt{s}}$, of which the first will be indepen-dent of u', and the second of them will be a function which will neither change sign nor magnitude for some values of u' equal and of contrary sign. The probability that these last two equations will hold simultaneously will be the product of their respective probabilities, as if these equations were two events independent of one another; because the probability of each of them being infinitely small, the existence of each equation is able to alter the probability of the other only by an infinitely small quantity of the second order. That being, if one eliminates x_1 between the two equations; if one makes, for brevity,

$$\frac{1}{s}\sum \lambda_n = m, \quad \frac{1}{s}\sum(\lambda_n - m)\sigma' = \lambda, \quad \frac{1}{2s}\sum(\lambda_n - m)^2 = \mu,$$

and if one neglects the square of σ', one will have

$$h = \mu + u\sigma - u'\lambda,$$

and the probability of this value of h will be infinitely small of the second order, namely:

$$\left(\frac{1}{\pi}e^{-u^2}e^{-u'^2} + uu'U' + u'uU\right)dudu',$$

by neglecting also the product UU' which is, by hypothesis, a quantity of the order of $\frac{1}{s}$.

I substitute this value of h into formula (11); I develop according to the powers of $u\sigma - u'\lambda$, of which I neglect the square which would be of the order of $\frac{1}{s}$; there comes

$$p = \frac{2}{\sqrt{\pi\mu}}\int_0^\delta e^{-\frac{v^2}{\mu}}dv + p'(u\sigma - u'\lambda);$$

p' being that which $\frac{dp}{du}$ becomes when one makes $h = \mu$ there.

This value of p would be the probability of the limits $x_1 \pm \frac{2\delta}{\sqrt{s}}$ of the mean result $\frac{1}{s}\sum\lambda_n$, if the value of h that one has substituted was certain; but the different values of h being only probables, the probability of these limits corresponding to each of these values, will be the product of the corresponding value of p, multiplied by the probability of that of h; consequently the total probability of these same limits, or their probability relative to all the values of h, will be the integral of this product, extended to all the values of u and u' which do not render insensible the coefficient of $dudu'$. After that, by neglecting always quantities of order $\frac{1}{s}$, and observing that the terms multiplied by an odd power of u or of u', vanish in the integrations, we will have

$$\frac{2}{\pi\sqrt{\pi\mu}}\int_0^\delta e^{-\frac{v^2}{\mu}}dv \iint e^{-u^2}e^{-u'^2}dudu',$$

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for the probability of which there is question; and as one is able, without sensible error, to extend the integrals relative to u and u' from $-\infty$ to $+\infty$, it will be reduced to $\frac{2}{\sqrt{\pi\mu}} \int_0^\delta e^{-\frac{v^2}{\mu}} dv$; this which is nothing other than formula (11), in which one has made $h = \mu$.

Thus, to the degree of approximation where we ourselves are stopped, that is to say by neglecting the quantities of order of $\frac{1}{s}$, the quantity μ is the value of h, which one must substitute into formula (11), or else into the limits of the mean result $\frac{1}{s}\sum\lambda_n$, to which corresponds formula (9). This value of h is able to be written under these two forms:

$$\left.\begin{aligned} h &= \frac{1}{2s}\sum(\lambda_n - m)^2 \\ h &= \frac{1}{2s^2}\sum\left[s\sum\lambda_n^2 - \left(\sum\lambda_n\right)^2\right], \end{aligned}\right\} \tag{12}$$

which are equivalent, by observing that one has made $\frac{1}{s}\sum\lambda_n = m$. The numeric calculation of the first expression will always be easy, according as the deviations of the observations on both sides of the mean, that is to say according to the values of $\lambda_n - m$; the calculation of the second will be generally much less convenient and often impractical.

Formula (11) and the value of h as function of the givens of the observation are due to Laplace, who has made a great number of interesting applications. Lagrange is the first who has submitted to analysis[1] the probability of the arithmetic mean among the observed results; but he has supposed known the law of probability of the values of the unknown; and it is to Laplace that one must have rendered the probability of the mean result independent of this law, in the case where the observations are in great numbers. The preceding analysis is proper, it seems to me, to dissipate the doubts which were able to remain yet on the use of the value of h and on the degree of exactitude of formula (11).[2]

8. The quantity x_1, toward which the mean result of the observations converges in measure as their number increases, is not necessarily one of the values of A which have greatest probability and are given most often by the isolated observations; it is even able to happen that its probability is completely null, so that this value of A is not able to be given by any observation in particular: it is this which will take place, for example, if all the functions $f_n x$ are nulls for one same value of x, and symmetric from one side and to the other. In the general case that we have considered, that is to say in the case where the curve of probability of which the equation is $y = f_n x$ changes from one observation to another, it is able yet to happen that the areas of all these curves did not have their centers of gravity on the same ordinate; then the abscissa x_1 will vary with the number s of observations; and if one divides s into two parts s' and s_1, which are still very large numbers, the mean results of these two partial series s' and s_1 of observations will not be the same, although the error to fear on each of them is very small, and although they had both a very large probability.

The calculation of the mean life is one of the most ingenious applications that one has made of the preceding formulas. We suppose that one considers a very great num-

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[1]Volume V of the old Mémoires de Turin.
[2]First Supplement to the *Théorie analytique des probabilités*, p. 7.

ber s, one million, for example, of infants born at one same epoch: if one designates by x any one time, and by $f_n x dx$ the infinitely small probability that one of these infants has of living the time x, and if one assimilates the duration of life to an eventual gain, the sum of all the possible values of x, multiplied by their respective probabilities, or $\int x f_n x dx$, will be the advantage of this infant, or his expectation of life. Consequently, the mean life will be the sum of these integrals relative to all the infants, divided by their number, or $\frac{1}{s} \sum \int x f_n x dx$; each integral being extended from $x = 0$ to a value of x which renders $f_n x$ null or insensible, and that one is able to regard as the limit of human life. Now, this quantity is that which we have designated by x_1; its approximate value will be therefore $\frac{1}{s} \sum \lambda_n$, in taking for λ_1, λ_2, etc., the ages to which the deaths are a number s of other individuals, born in the same country as the infants that one considers, and in an epoch as near as it will be possible with the birth of those. The same values of λ_1, λ_2, etc., will serve to calculate the probability that the difference $x_1 - \frac{1}{s} \sum \lambda_n$, or the error of $x_1 = \frac{1}{s} \sum \lambda_n$, is contained between some given limits. The unknown function $f_n x$ is very different for the different infants who are born at the same epoch and in one same country; but the mean function $\frac{1}{s} \sum f_n x$, and hence the mean life $\frac{1}{s} \sum \int x f_n x dx$, varies without doubt only with slowness, through the extinction of the maladies and the perfection of society. Experience alone is able to teach us if this duration of mean life is stationary, or if it changes sensibly in some great intervals of time.

It is by the same principles that one calculates the mean benefit and its probability, that one is able to expect from a very great number of speculations, according to the gains and losses known of another very great number of similar operations, that is to say of which the mean probability is supposed the same.

9. When one proposes to determine, by a series of observations, the magnitude of a phenomenon or the measure of any one thing A, one supposes implicitly that among all the values of which A susceptible *a priori*, there exists one of them such that it is equally probable that one of them will deviate equally on each side in each observation; one supposes moreover that this unknown value is the same for all the observations that one is going to make; and it is this value of A that one wishes to find. This comes back to say that all the curves which are deduced from the equation $y = f_{nx}$ are symmetric on both sides of one of their points, and that this point corresponds to the same abscissa for these different curves, which abscissa represents the unknown value of A. Under this hypothesis, the centers of gravity of their areas, and the one of the area of the mean curve, of which the equation is $y = \frac{1}{s} \sum f_n x$, will be situated on a common ordinate, of which the abscissa will be this same value. By multiplying the observations, the quantity x_1 of which one will approach indefinitely, will be constant or independent of their number s; and it will have the probability p, given by formula (9), that their mean result $\frac{1}{s} \sum \lambda_n$ will not deviate from x_1 or from the true value of A, by a quantity greater or smaller than $\frac{2 \gamma \sqrt{h}}{\sqrt{s}}$. The value of h will be given also by the observations, as one has seen above; it will depend on their degree of precision; and if there is question, for example, of the measure of an angle, that quantity h will be able to be very different for two series of observations made with some instruments or by some different observers. If there is a question of the magnitude of a phenomenon, as, for example, the difference of the heights of the barometer at two epochs determined from the day, h will depend again on some accidental and variable causes which influence unequally on these heights,

and which one is able to attribute to the state of the atmosphere.

But, as small as the limit $\frac{2\gamma\sqrt{h}}{\sqrt{s}}$ of the error to fear be, by taking $\frac{1}{s}\sum\lambda_n$ for the value of A, and as probable as this limit be, one must not lose from view that this value is subordinate to the hypothesis that one has made, of the symmetry of all the functions $f_n x$ on both sides of one same value of x. If some unknown cause renders dominating, either in on sense, or in an opposed sense, the errors of the instruments, or the variable circumstances which influence the phenomena, or else further, if the magnitude of A varies during the duration of the observations, he hypothesis of which there is question will not have place: the quantity $\frac{1}{s}\sum\lambda_n$ will always be the approximate value of the abscissa x_1; but x_1 will no longer represent the thing that one wished to determine, and the observations ought be rejected. It would therefore be important to understand, by the observations themselves, if they are incompatible with the hypothesis of the symmetry of $f_n x$; or, there exists, in effect, some conditions to which they must satisfy, if this hypothesis is applicable to the laws of probability of the values of A.

10. We will arrive to some similar conditions, by taking for the function X an odd power of $x - x_1$, that is to say by designating by i a positive and odd number, and making

$$X = (x - x_1)^i.$$

According to the notations of n° 7, the quantities k_n and k'_n of n° 4 will be

$$k_n = \int_{a'}^{b'} z^i f'_n z\,dz, \quad k'_n = \int_{a'}^{b'} z^{2i} f'_n z\,dz.$$

Under the hypothesis of all the functions $f_n z$ symmetric on both sides of one same value of x, this value will be x_1, and one will have

$$f'_n z = f'_n(-z), \quad a' = b';$$

this which will render null the value of k_n. The quantities k and h of n° 4 will be then

$$k = 0, \quad h = \frac{1}{s}\sum\int_0^{b'} z^{2i} f'z\,dz.$$

According to n° 5, there will be therefore the probability p given by formula (9), that $\sum(\lambda_n - x_1)^i$ will be less than $2\gamma\sqrt{hs}$, setting aside the sign. This probability will be equal to $\frac{1}{2}$, for example, by taking $\gamma = 0.47614$. But the number s of observations being very great, it is very probable that their mean result $\frac{1}{s}\sum\lambda_n$ will differ very little from x_1, and that at the same time the sum $\sum(\lambda_n - x_1)^{2i}$ will be very near the value of $\sum\int_{-b'}^{b'} z^{2i} f'z\,dz$, or $2hs$. Therefore by making, for brevity,

$$\frac{1}{2}\sum\lambda_n = m, \quad \frac{\sum(\lambda_n - m)^i}{\sum(\lambda_n - m)^{2i}} = r,$$

there will be a probability very little different from p, that this ratio r will be smaller than $\gamma\sqrt{2}$; and by taking for γ the value which gives $p = \frac{1}{2}$, one will be able to wager, very nearly, one against one, that one will have

$$r < (0.47614)\sqrt{2}, \quad \text{or} \quad r < 0.67336,$$

if the hypothesis $f'z = f'(-z)$ really holds. Consequently, if one calculates the ratio r for a determined exponent, and if one finds its value superior to 0.67336, or even a little under this fraction, that will suffice in order that the hypothesis $f'z = f'(-z)$ not be probable, and in order that one must, consequently, reject these observations as improper to make known the true value of A that one wishes to find.

11. In a great number of cases, and especially in the questions of Astronomy, the quantity that one proposes to determine by the observations is a given function of many elements which are already known by approximation, and to which there is no more question but to make undergo some very small corrections, of which one neglects the products and the powers superior to the first. The given function must then be a linear function of these unknown corrections: one equates it successively to all the values resulting from experience, this which furnishes as many equations of condition as one has observations. The usage of these linear equations in order to determine the corrections of the elements by making them to unite a great number of observations, has contributed much to the perfection of astronomical tables. It appears that Euler and Mayer are the first who have employed them, one in his Memoire on the libration of the Moon, and the other in his piece on the Perturbations of Jupiter and of Saturn, crowned by our Academy in 1750. But their number always being superior to the one of the unknowns, one was embarrassed to resolve them, and there resulted from it this grave inconvenience, that the calculators were able to deduce from one system of equations, different results, according to the method of calculation that they employed. This embarrassment has subsisted until the epoch when Mr. Legendre proposed a direct and uniform method, which was generally adopted under the name of *method of least squares of errors*, which his author has given to it. It consists, as one knows, in subtracting from the result of each observation, the linear function of which it furnishes an approximate value: the difference is the error of the observation; one makes the sum of the squares of all these differences, next one equates to zero these differentials, taken successively with respect to the corrections of all the elements; this which gives as many equations as one has unknowns to determine. This method, had it only the advantage of uniformity and of freeing the process of the calculation from all indetermination, would be already an important service that our illustrious colleague has rendered to the sciences of observation; but it is further that which allows to fear the *minimum* of error on the value of each element, thus as Laplace has proven by the calculus of probabilities. We add, in terminating this Memoire, that after having calculated the corrections of the element by the method of least squares, and having substituted their values into the linear expressions of the errors of the observations, if one made the sum of the odd powers of all the errors, and if one divided it by the square root of the sum of the their double powers, the magnitude of the quotient will furnish a *criterium*, according to which one must reject the observations, or to adopt the results of them, if they have besides a sufficient probability. One would find, in effect, that it is very probable that this quotient must be a fraction of little consequence; and by a rather complicated calculation, one would be able to determine, whatever be the number of elements corrected, the exact value of this fraction for a determined degree of probability.