ON WALDEGRAVE'S PROBLEM FROM THE CORRESPONDENCE OF MONTMORT AND NICOLAS BERNOULLI

EXTRACTED FROM THE
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
2ND EDITION OF 1713

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli
At Montmort 10 April 1711 (pages 318–320)

. . . This same Geometer[1] who is a Gentleman of much intellect, has proposed to me lately & has resolved a quite pleasing Problem which is here. *Pierre, Paul & Jacques play a pool at Trictrac or at Piquet. After one has deduced whom will play it is found that Pierre & Paul begin. We demand, 1°. What is the advantage of Jacques. 2°. How great are the odds that Pierre or Paul will win rather than Jacques. 3°. How many games must the pool naturally endure.*

As you do not know perhaps what it is to play a pool, I am going to explain it to you, nothing is more simple. If Pierre wins, Jacques will enter in the place of Paul & will put an écu into the pool; then if Pierre wins, the pool is ended, & Pierre wins two écus. If Jacques wins, Paul enters in the place of Pierre. In a word the one who enters always puts an écu into the game, & the one who wins two games in sequence takes away all that which is in the pool. If there were four Players, it would be necessary to win three games in sequence; & four if there were five Players. I have found that to three Players the advantage of Jacques, naming a the stake of each Player, was contained in this series

$$\frac{3}{2^2}a + \frac{5a}{2^5} + \frac{7a}{2^8} + \frac{9a}{2^{11}} + \frac{11a}{2^{14}} + \&\text{c.}$$
$$- \frac{a}{2} - \frac{a}{2^3} - \frac{2a}{2^4} - \frac{2a}{2^6} - \frac{3a}{2^7} - \frac{3a}{2^9} - \frac{4a}{2^{10}} - \frac{4a}{2^{12}} - \frac{5a}{2^{13}} - \&\text{c.}$$

that which is reduced to this simpler series;

$$\begin{aligned} &\frac{a}{2^3} + \frac{\text{zero}}{2^6} - \frac{a}{2^9} - \frac{2a}{2^{12}} - \frac{3a}{2^{15}} - \frac{4a}{2^{18}} - \&\text{c.} \\ =&\frac{a}{8} - \frac{1}{8^2} \times \frac{a}{8} + \frac{2a}{8^2} + \frac{3a}{8^3} + \frac{4a}{8^4} + \&\text{c.} \\ =&\frac{a}{8} - \frac{a}{8^2} \times \overline{m + 2mm + 3m^3 + 4m^4 + 5m^5}, \end{aligned}$$

by supposing $m = \frac{1}{8}$. Now in order to find the sum of this series $m + 2mm + 3m^3 + 4m^4 + \&\text{c.}$ where the coefficients & the exponents are in arithmetic progression, I observe

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[1] Mr. de Waldegrave

that

$$\begin{aligned}
\tfrac{m}{1-m} &= m + mm + m^3 + m^4 + m^5 \\
\tfrac{mm}{1-m} &= \qquad mm + m^3 + m^4 + m^5 \\
\tfrac{m^3}{1-m} &= \qquad\qquad m^3 + m^4 + m^5 \\
\tfrac{m^4}{1-m} &= \qquad\qquad\qquad m^4 + m^5
\end{aligned}$$

Whence I conclude that the sought sum is equal to this one

$$= \frac{m}{1-m} + \frac{mm}{1-m} + \frac{m^3}{1-m} + \frac{m^4}{1-m}\text{\&c.} = \frac{m}{\overline{1-m}^2},$$

and consequently the advantage of Jacques $\frac{6}{49}$. I have further found that although there is the advantage for Jacques, there are odds five against 4 that Pierre will win the pool rather than Jacques.

If we wish to know how long the pool will endure among three Players, we will find that there are odds three against one that it will endure no more than three games, 7 against 1, 15 against 1, 31 against 1, that it will not endure more than 5, 7, 9, games; I have similarly sought how long the pool would endure among four Players, & I have found this sequence

$$\frac{1}{4}, \frac{3}{8}, \frac{8}{16}, \frac{19}{32}, \frac{43}{64}, \frac{94}{128}, \frac{201}{256}, \frac{423}{512}, \frac{880}{1024}, \frac{1815}{2048}, \frac{3719}{4096}, \text{\&c.}$$

of which the sequence was not easy to notice. I have wished to seek the lot of the Players when there are four, & also how long the pool will endure when there are five or six Players; but this has appeared to me too difficult, or rather I have lacked courage, because I would be sure to succeed to it. I propose to the Geometers the solution of the Problem on the Lottery of Lorraine. I invite you, Sir, to render public that which you have found. As there remains nearly no more copies of my Book, I believe that I will have to give soon a new edition of it. When I will be determined, I will demand permission from you, & to Mr. your Uncle, to insert your good Letters which will be the principle ornament of it. One counsels me to change the order & form of it, & to reassemble in the first Part all the Theory of Combinations. I have similar design to give the demonstrations of quantities of propositions & of difficult solutions that I have omitted by design in the first edition. You obligate me much, Sir, to give me your opinion on this subject.

Lottery of Lorraine

It is not it seems to me of these demonstrations as of the demonstrations of Geometry, those touching the numbers & combinations are infinitely more embarrassing, & one is able to have them very sharply in the mind with being able to set them on paper. You arrange in series content, for example, of the demonstration which is for proposition 14, page 97.[2] You make me too much honor, Sir, to believe me capable to fulfill the views that the late Mr. your Uncle had, to treat by Geometry the things of politics & morals. For me the more I touch & the more I recognize my insufficiency in this regard: I have some ideas & some materials, but it is yet little of things. The concern is to discover the truths of practice & in the usage of civil society. It is necessary to be based on some exact & well established hypotheses, to conserve especially this exactitude of which the Geometers are stung more than the rest of men, all that demands a strong head & a very great work. I have read lately a quite beautiful morsel of Mr. your Uncle in the Memoires de l'Academie de Berlin. I am astonished to see the Journals of Leipzig so stripped of morsels of Mathematics: they owe their reputation in part to the excellent Memoirs that Messers your Uncles sent often: the Geometers no longer find five or six years the same riches as othertimes, make some

[2]See page 44.

reproaches to Mr. your Uncle, & permit me to make of your also, *Luceat lux vestra coram hominibus.*[3] I am, &c.

Extract of the letter from Mr. Nicolas Bernoulli to Mr. de Montmort
At Basel 10 November 1711 (pages 328–331)

...But for this other Problem that this Geometer has proposed to you, I have resolved it generally in all its three points. Let be named n the number of matches which it is necessary to win in sequence, or the number of Players less one; Pierre & Paul two Players who follow immediately in the order of play; so that Paul, for example, enters into the game immediately after Pierre; a the probability that Pierre will win the pool; b the probability that Paul will win it; A the advantage or the disadvantage of Pierre; B the advantage or disadvantage of Paul, I find generally $b = \frac{a\times 2^n}{1+2^n}$, & $B = \frac{\overline{A+a}\times 2^n - nb}{1+2^n}$. The first of these two equations demonstrate that there are odds of $1+2^n$ against 2^n that Pierre will win the pool rather than Paul, that which gives in the particular case of $n=2$, five against four, thus as you have found. It is easy to find by these two equations or Theorems, the advantage or the disadvantage of each Player, & the probability that each has to win the pool, because the sum of the advantages & of the disadvantages of all the Players together must be equal to zero; as also the sum of the probabilities which they have to win the pool, must make an entire certitude or 1. For example in the case of the three Players or of $n = 2$, we find first that the probability which the first & second have to win, that is to say those who play first, is $= \frac{5}{14}$, & that what the third or the one who enters into the game last is $= \frac{2}{7}$, having substituted these values into the second equation, & having named x the advantage or the disadvantage of the first, that of the second will also be$= x$, & the one of the third $= \frac{\overline{x+\frac{5}{14}}\times 4 - 2\times\frac{2}{7}}{5} = \frac{4x+\frac{6}{7}}{5}$, to which, if we add $2x$, we will have $\frac{14x+\frac{6}{7}}{5} = 0$; whence we deduce $x = -\frac{3}{49}$, that which shows that the first two players have disadvantage, & that the advantage of the third is $= \frac{6}{49}$, thus as you have found it by a very different way from that. I have also made application for the case of four & five Players, & I have found that in a pool of 4 Players the disadvantage of the first two is $= -\frac{2700}{22201}$, the advantage of the third $= \frac{1176}{22201}$, & the advantage of the fourth $= \frac{4224}{22201}$; but to five Players the disadvantage of the first two is $= -\frac{24059828}{131079601}$, the advantage of the 3rd $= \frac{-2402712}{131079601}$, the advantage of the fourth $= \frac{16789760}{131079601}$, & the one of the fifth $= \frac{33732608}{131079601}$. For the last point of this Problem, namely how many games must the pool naturally endure, I have found a general formula which expresses the probability that that it will be decided in at least p games, here it is

$$\frac{p+1}{1.2^n} - \frac{p-n.p-n+3}{1.2.2^{2n}} + \frac{p-2n.p-2n+1.p-2n+5}{1.2.3.2^{3n}} - \frac{p-3n.p-3n+1.p-3n+2.p-3n+7}{1.2.3.4.2^{4n}} + \frac{p-4n.p-4n+1.p-4n+2.p-4n+3.p-4n+9}{1.2.3.4.5.2^{5n}} - \&\text{c.}$$

[4] It is necessary to take as many terms of this series as there are units in $\frac{p+n}{n}$. Now if you prefer the series such as you have given for three & four Players, here is a general method to find them. It is necessary to construct a series of fractions, of which the denominators increase in double ratio, & in which the first term is $\frac{1}{2}$ raised to $n-1$, that is to say, to the

[3]Let your light so shine among men. Matthew, 5:16.

[4]*Note*. The numbers formed in this manner have many very singular properties, Mr. de Cassini has given some of them, & has applied them to the Theory of the Planets.

exponent expressed by the number of Players less 2, & the numerator of each other term the sum of the numerators of as many preceding terms as there are units in $n-1$. This being made, the sums of the terms of this series will give the terms of the sought series; namely, the first term will be also the first of the sought series, the sum of the first two will be the second term, the sum of the first three will be the third term, that of the first four the fourth, & thus in sequence. By this manner we will find for five Players this sequence $\frac{1}{8}, \frac{3}{16}, \frac{8}{32}, \frac{20}{64}, \frac{47}{128}, \frac{107}{256}, \frac{238}{512}, \frac{520}{1024}, \frac{1121}{2048}, \frac{2391}{4096}$,&c. of which the terms are the sums of these $\frac{1}{8}, \frac{1}{16}, \frac{2}{32}, \frac{4}{64}, \frac{7}{128}, \frac{13}{256}, \frac{24}{512}, \frac{44}{1024}, \frac{81}{2048}, \frac{149}{4096}$, &c. in which each numerator is the sum of the three preceding. I am astonished that by giving you two series for the case of three & of four Players you have not observed the progression which there is between these series, & which on the contrary it has appeared too difficult or too painful to you to continue them for a great number of Players; but I am not astonished that you have found the same difficulty in wishing to seek the lot of the Players, when there are more than three, because it is extremely difficult to find it by the way of infinite series, as you have done for the case of three Players; & if I had not found a method to resolve this Problem by analysis, it would have been absolutely impossible to me to succeed to it. I would have well wished to make you part of this method; but as it would be too lengthy to put it here, & as I would have the pain to make myself well understood, I will leave to you the pleasure of finding it for yourself.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli
At Paris 1 March 1712 (pages 345–346)

Your method to find the advantages or disadvantages of those who play a pool at Trictrac, at the rate of the order according to which they enter the game, can be only perfectly beautiful. This Problem is assuredly quite difficult. I have wished to discover how we can apply your method for three Players to the case of four or five Players, but uselessly. The route that you have followed is apparently very remote. I will work seriously as soon as I will have the leisure. Your series to determine how many matches the pool must naturally endure is quite correct.

Extract of the letter from Mr. Nicolas Bernoulli to Mr. de Montmort
At Basel 2 June 1712 (pages 350–351)

For that which is my method to find the advantages or disadvantages of those who play a pool, I have believed to have explained it quite clearly, & I am bothered that you have not been able to apply it to the case of four or of five Players; I am going therefore to clarify it more for you by applying the two Theorems which I have found in the case of the four Players. Let the four Players be Pierre, Paul, Jacques & Jean, who enter into the game according to the order which they are ranked here; so that Pierre & Paul play first together; next the one who will have won will play with Jacques, & the one who will have won of these two there with Jean, & thus in sequence; let be named p the probability that Pierre or Paul has to win the pool, q the probability that Jacques has to win, & r the probability that Jean has to win it, x the advantage of Pierre or Paul, y the advantage of Jacques, & z the advantage of Jean, we will have by supposing $n = 3$ which is the number of games which it is necessary to win in sequence; by the first Theorem $q = \frac{p\times 2^3}{1+2^3}$, $r = \frac{q\times 2^3}{1+2^3}$, & by the second Theorem $y = \frac{\overline{x+p}\times 2^3-3q}{1+2^3}$, & $z = \frac{\overline{y+q}\times 2^3-3r}{1+2^3}$; now $p+p+q+r$ must be $= 1$, & $x+x+y+z = 0$; we will have therefore these six equations $q = \frac{8p}{9}$, $r = \frac{8q}{9}$, $y = \frac{8x+8p-3q}{9}$, $z = \frac{8y+8q-3r}{9}$, $2p+q+r = 1$, & $2x+y+z = 0$, which being compared together by the ordinary methods, gives $x = -\frac{2700}{22201}$, $y = \frac{1176}{22201}$, & $z = \frac{4224}{22201}$. The route which I have followed in order to find these two Theorems is not at all remote, I would communicate it to you willingly if I were not so pressed, that which is also the cause that I pass under silence the other places of your Letter. I know nothing at all new of the sciences, except that Mr. de Moivre who is member of the Society in England, made imprinted at London a Book on Chances. As I believe that you will be curious to have this Book when it will be imprinted, & that I hope to pass from Holland into England, I will try to procure a copy.

Extract of the letter from Mr. de Montmort to Mr. Nicolas Bernoulli
At Montmort this 5 September 1712 (pages 362 & 366)

I have received at the beginning of the month of August the Book of Mr. Moivre, the author has addressed it for me to Mr. Abbé Bignon who has had the goodness to send it to me. On that which you have sent to me, & on the manner in which the Author speaks in the Preface, I expected an entirely other thing; I hoped to find the solution of the four Problems which I proposed at the end of my Book, or at least the solution of any one of the four, & some novelties of this kind fit to extend the route which I have opened; but you will find that his work is limited almost entirely to resolve in a more general manner what I have done, the most simple & most easy questions which are in my Book; for example, the five Problems of Mr. Huygens which I have treated only summarily because of their extreme easiness, in comparison to most of the other Problems which are resolved in my Book. You will find finally that the questions which he treats, which are not at all resolved, are in our Letters; so that I do not believe that he has in this Work, in other respects very good, nothing new for us, & nothing which can please us by singularity, if this is not the manner to find which is often new, & always good & ingenious. Here are some remarks that I have cast in haste on paper these past days, when I worked to render account of this Work to Mr. Abbé Bignon who has demanded of me my sentiment. You know without doubt that this illustrious Abbé, who is in France the Protector of the Sciences & the Scholars, has an extent of knowledge well beyond ordinary limits, a very great passion for all that is from activity of the mind, & much ardor to contribute to the perfection of the sciences.

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Problem 15 is our Problem of the Pool of which I have sent you the solution in my Letter of 10 April 1711, I have been quite surprised to find this quite hazardous Corollary of the Author: *Si plures sint collusores, ratio sortium eadem ratiocinatione invenietur.* You have made me think, Sir, that the application of this Problem to the case of four & of five & of six Players was infinitely more difficult than that which is limited to three Players. The way of infinite series that Mr. Moivre employed, & which is also employed in my Letter, is easy for three Players, but absolutely impractical for many Players.

Extract of the letter from Mr. Nicolas Bernoulli to Mr. de Montmort
At London this 11 October 1712 (page 375)

I have the pleasure to see here often Mr. de Moivre who has made a present to me of his Book *De Mensura Sortis*. He has said to me the he has sent to you also a Copy, & he awaits with impatience your sentiment on this Work. You will be astonished to find many of the Problems which we have resolved, & among others also the one of the duration of games by reduction, which he has resolved in a manner, quite different from ours, nonetheless very good & very curious. He has also resolved the Problem of the Pool for three Players by the way of infinite series, & he has advanced in a Corollary, that if the number of the Players were greater, one could find their lots by the same reasoning. As I have shown you the impossibility that there is in succeeding by this method of infinite series, I believe that you will make of this Corollary the same judgment that I have made of it. I have communicated to him the two Theorems which I have found, after having made him sense the difficulty to use his method, when the number of Players is greater than three.

Extract of the letter from Mr. Nicolas Bernoulli to Mr. de Montmort
At Brussels this 30 December 1712 (page 380–387)

As you greatly wish to see my method & my demonstrations for the pool, I am going to report to you here all at length. I have differed this response a little from what I would have sent you otherwise from Holland, in order to give me the leisure to recall my ideas, & to put me in a state to content you entirely. This is that solution of the three Problems which you have proposed on the pool which I prefer to everything which I have found until now in these matters. Here are the reasonings which I have made in order to succeed at these three Problems; I offer them to you methodically & all at length in the two Tables following, in order to render me more intelligible.

PROBLEM I.

Many players of whom the number is $n+1$ play a pool,we demand what is the probability that each has to win the pool.

SOLUTION. Let be called t the expectation to win that one of the two who enter first into the game has, u the expectation that the one who enters second into the game has, x the expectation of the third, y that of the fourth, z that of the fifth, &c. Let moreover be called a the expectation to win the pool which a Player who enters into the game has, & who plays against one who has not yet won some matches; b the expectation of the one who enters into the game & who plays against one who just won one match; c the expectation of the one who plays against one who just won two matches; d the expectation of the one who plays against one who just won three matches, &c. Let further p be called the lot or expectation of the one who exits from the game leaving a Player who has won one match; q the lot of the one who exits from he game leaving a Player who has two matches; r the lot of the one who exits from the game leaving a Player who has three matches; &c. Thus put we will have the following equations marked No. 1, No. 2, No. 3, &c. up to No. 10 in Table I.[5]

$$\begin{array}{lcccccccccccc} & & & & & & & & & & & & \text{No. 1}\\ & t & + & t & + & v & + & x & + & y & + & z & +\&c. \quad =1\\ \text{No. 2} & \| & & \| & & \| & & \| & & \| & & \| & \\ & a & & a & & b & & \frac{1}{2}c+\frac{1}{2}b & & \frac{1}{4}d+\frac{1}{4}c+\frac{1}{2}b & & \frac{1}{8}e+\frac{1}{8}d+\frac{1}{4}c+\frac{1}{2}b & \&c.\end{array}$$

The equation marked No. 1, is evident; because the lots or the expectations of all the Players taken together must make 1 or an entire certitude: the other equations are found in the manner that I just explained. Among the equations marked No. 2, we find, for example, $z=\frac{1}{8}e+\frac{1}{8}d+\frac{1}{4}c+\frac{1}{2}b$; because the one who enters the fifth into the game will play against one who will have won either 4, or 3, or 2, or 1 match; now there are odds $\frac{2}{16}$ or $\frac{1}{8}$ that one or the other of the first two Players win four games in sequence; & $\frac{1}{8}$ of probability that he

[5]*Translator's note*: I have divided Table I and inserted the various Equations as appropriate into the text. Table II, however, is intact.

will play against one who has won three matches, $\frac{1}{4}$ that he will play against one who has won two matches, & $\frac{1}{2}$ that he will play against one who has won one match; therefore his lot or $z = \frac{1}{8}e + \frac{1}{8}d + \frac{1}{4}c + \frac{1}{2}b$.

Among the equations No. 3, we find, for example $c = \frac{1}{2}r + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1$; because there are odds of $\frac{1}{2}$ that a Player who newly enters into the game will not win any match, $\frac{1}{4}$ that he will win only one, $\frac{1}{8}$ that he will win only two, $\frac{1}{16}$ that he will win only three, &c. $\frac{1}{2^n}$ that he will win all the games which he must less 1, and further $\frac{1}{2^n}$ that he will win all the games which he must; if he wins none of them, it leaves a Player who has won three matches, since we suppose in this example that he plays against a Player who has already won two matches; if he wins some of them, but not all that he must for himself, he exits the game, leaving a Player who has won one match; therefore his lot or c is $\frac{1}{2}r + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1$.

$$\left.\begin{array}{c|c|l}
\multicolumn{3}{l}{\text{Enter}} \\
0 & a & a = \overline{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
1 & b & b = \frac{1}{2} \times q + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
2 & c & c = \frac{1}{2} \times r + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
3 & d & d = \frac{1}{2} \times s + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times p + \frac{1}{2^n} \times 1 \\
4 & e & \\
\vdots & \vdots &
\end{array}\right\} \text{No. 3}$$

Equations, No. 4, are found by a parallel reasoning; because a Player who exits from the game leaving, for example, a Player who has won one match, acquires the expectation either of the one who enters second into the game, or of the one who enters third, or of the one who enters fourth, &c. according as the Player who has left in the game won either one, or two, or three, &c. matches less than it is necessary for him to win the pool.

$$\left.\begin{array}{cll}
\multicolumn{3}{l}{\text{Exit}} \\
1 & p & p = \frac{1}{2^{n-1}} \times v + \frac{1}{2^{n-2}} \times x + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
2 & q & q = \frac{1}{2^{n-2}} \times x + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
3 & r & r = \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots \\
4 & s & s = \frac{1}{2^{n-4}} \times z + \cdots \\
\vdots & \vdots &
\end{array}\right\} \text{No. 4}$$

Equations No. 5, are found by the subtraction of equations No. 3; & those No. 6, by the subtraction of equations No. 4. Equations, No. 7, are found by substituting into equations, No. 5, the values found in equations No. 6.

$$\left.\begin{array}{ccccccc|l}
a - b & = & \frac{1}{2}p - \frac{1}{2}q & = & \frac{1}{2^n} \times v & = & t - v & p - q = \frac{1}{2^{n-1}} \times v \\
b - c & = & \frac{1}{2}q - \frac{1}{2}r & = & \frac{1}{2^{n-1}} \times x & = & 2v - 2x & q - r = \frac{1}{2^{n-2}} \times x \\
c - d & = & \frac{1}{2}r - \frac{1}{2}s & = & \frac{1}{2^{n-2}} \times y & = & 4x - 4y & r - s = \frac{1}{2^{n-3}} \times y \\
 & \text{No. 5} & & \text{No. 7} & & \text{No. 9} & &
\end{array}\right\} \text{No. 6}$$

Equations, No. 8, are found by seeking the values of $a,\ b,\ c,\ d,$ &c. by equations No. 2; & these values being substituted into equations No. 5, we will have equations No. 9, which were compared with equations No. 7, give equations No. 10, & these last equations furnish my first Theorem.

$$
\begin{array}{l|l}
\multicolumn{1}{c}{\text{No. 8}} & \\
a = t & v = t \times \frac{2^n}{1+2^n} \\
b = v & x = v \times \frac{2^n}{1+2^n} \\
c = 2x - b = 2x - u & y = x \times \frac{2^n}{1+2^n} \\
d = 4y - c - 2b = 4y - 2x - v &
\end{array}
\left.\vphantom{\begin{array}{l}a\\b\\c\\d\\e\end{array}}\right\}\text{No. 10}
$$

PROBLEM II.

Being posed that which in the preceding Problem we demand what is the advantage or the disadvantage of each Player.

SOLUTION. As I am being served by the letters a, b, c, d, &c. p, q, r, s, &c. t, u, x, y, z, &c. to express the different probabilities that the Players have to win according to the different states in which they can be found; thus I will be served by similar capital letters A, B, C, D, &c. P, Q, R, S, &c. T, U, X, Y, Z, &c. to express the portion that each Player can claim in these different states; I suppose also that one puts into the game only when one just lost against a Player, & I call this stake 1. By following some reasonings similar to those which we have made in the preceding Problem, we will have the equations marked No. 1, No. 2, &c. up to No. 13 of the 2nd Table. In the equations No. 2, we have, for example $Y = \frac{1}{4}D + \frac{1}{4} \times \overline{C + c} + \frac{1}{2} \times \overline{B + 2b}$; because if the one who enters fourth into the game is obliged to play against one who has won three matches, his expectation is D; if he is obliged to play against one who has won two matches, his expectation is $C + c$; I add c to C, because the one who enters fourth finds three écus put into the game, instead that C is the expectation of the one who players against a Player who has two matches under the supposition that he has only two écus into the game; because the letters A, B, C, D, E, &c. P, Q, R, S, &c. signify the lots of the Players in the first entrances and exits; it is necessary therefore to add to C the portion which he can claim of that écu of surplus; now as in this state the probability to win the pool or to win this écu, is c, this portion will be $c \times 1$. Thus if he plays against a Player who has won only one match, it is necessary to add to the expectation B yet $2b$, because he finds two écus more in the game than the one finds who plays against a Player who has won one match, & the probability of winning these two écus is b, & consequently the portion which he can claim is $b \times 2$. The reasoning which we make for equations No. 3 & No. 4 is immediately parallel to the one which we just made for those of No. 2, & which we have made for equations No. 3 & No. 4 of the preceding Problem. Equations, No. 5 and No. 6, are found as in the preceding Problem. Equations, No. 7, are found by substituting the first equation of No. 3, Table I, into equations No. 5. Equations, No. 8, are found by substituting the 1st equation of No. 4, Table I, into equations No. 6. Equations No. 9, are found by substituting equations No. 8 into equations No. 7. Equations No. 10, are found by seeking the values of A, B, C, D, &c. through equations No. 2 of Table I & 2, or No. 2 of the 2nd, & No. 8 of the 1st Table; & these values being substituted into the equations No. 5, we have equations No. 11, which compared with equations No. 9, form equations No. 12; & these equations No. 12, compared with equations No. 10 of Table I, give equations No. 13, which furnishes my second Theorem.

PROBLEM III.

Being put that which above, we demand what is the probability that the pool will be won precisely after a number of given matches.

SOLUTION. Let be expressed by this sequence of letters $a,\ b,\ c,\ d,\ e,\ f,$ &c. the probabilities that the pool will be ended precisely at $n,\ n+1,\ n+2,\ n+3,$ &c. matches; it is evident that there must be at least n matches, since it is necessary to win n matches in sequence, & that the probability so that one of the first two Players wins first the n matches is $\frac{1}{2^{n-1}}$; because there are odds for each of the first two Players of $\frac{1}{2^n}$; therefore a will be $=\frac{1}{2^{n-1}}$. The values of the other letters are found always in a like manner; for example, the sixth term f is equal to $\frac{1}{2}e+\frac{1}{4}d+\frac{1}{8}c+\frac{1}{16}b+$&c. where it is necessary always to take as many preceding terms as there are units in $n-1$; whence it follows that the first term being given, we will have all the following. I will give the demonstration of it in a particular example, because this will be the same for all the other cases. Let, for example, the number of Players be $=5$, we demand what is the probability that the game will end in precisely ten matches. It is evident that the one who must win the pool at the 10th match, must enter into the game after the 6th match, & that he must win four matches in sequence. Now he can enter into the game finding a Player who has won either 1, or 2, or 3 matches; if he finds a Player who has won one match, there are as much odds that the pool will be decided at the 9th match, as there are odds after he has vanquished his adversary as he will win it himself at the 10th match; therefore before he has beat his adversary the probability that he will win the pool will be the half of this expectation, that is to say $\frac{1}{2}f$ (I call $g,\ f,\ e,\ d,\ c,$ &c. the probability that one will win the pool at the 10th, 9th, 8th, 7th, &c. match). If he finds by entering into the game a Player who has two matches, there are as much odds that his adversary will win at the 8th match, as there are odds after he will have vanquished two of his adversaries, that he will win himself at the 10th match; now as the probability that he beats two Players in sequence is $\frac{1}{4}$, the probability that he will win the pool at the 10th match, will be in this case $=\frac{1}{4}e$. If by entering into the game he finds a Player who has won 3 matches, there is as much odds that his adversary will win at the 7th match as he will have odds after having beat his first three adversaries, he will win yet the 4th; now the odds are $\frac{1}{8}$ that he will beat three adversaries in sequence; therefore his probability to win the pool in the 10th match in this case, will be $\frac{1}{8}d$; therefore all these three probabilities taken together are $g=\frac{1}{2}f+\frac{1}{4}e+\frac{1}{8}d$; *That which is was necessary to demonstrate.* We demonstrate in the same manner that $f=\frac{1}{2}e+\frac{1}{4}d+\frac{1}{8}c$; $e=\frac{1}{2}d+\frac{1}{4}c+\frac{1}{8}b$; $d=\frac{1}{2}c+\frac{1}{4}b+\frac{1}{8}a$; $c=\frac{1}{2}b+\frac{1}{4}a$; $b=\frac{1}{2}a$. Now a in this case of five Players is $=\frac{1}{2^{n-1}}=\frac{1}{8}$; therefore $b=\frac{1}{16}$, $c=\frac{2}{32}=\frac{1}{16}$, $d=\frac{4}{64}=\frac{1}{16}$, $e=\frac{7}{128}$, $f=\frac{13}{256}$, $g=\frac{24}{512}=\frac{3}{64}$, &c. *That which it was necessary to find.* The sum of as many of these terms $a,\ b,\ c,\ d,\ e,\ f,$ &c. as there are units in p, will express the probability that the pool will be ended in at least $n+p-1$ matches. If we wish to have a formula to express this sum, we will have by putting p for the number of terms

$$\frac{p+1}{1.2^n}-\frac{p-n.p-n+3}{1.2.2^{2n}}+\frac{p-2n.p-2n+1.p-2n+5}{1.2.3.2^{3n}}\\ -\frac{p-3n.p-3n+1.p-3n+2.p-3n+7}{1.2.3.4.2^{4n}}+\text{\&c.}$$

for the expression of this formula; & the formula to express any term of this series $a,\ b,\ c,\ d,\ e,$ &c. of which the quantity is p, will be

$$\frac{1}{2^n}-\frac{p-n+1}{1.2^{2n}}+\frac{p-2n.p-2n+3}{1.2.2^{3n}}-\frac{p-3n.p-3n+1.p-3n+5}{1.2.3.2^{4n}}+\text{\&c.}$$

We will find easily the demonstration of these formulas, by supposing that the numerator of each term of this series $a,\ b,\ c,\ d,$ &c. is the sum of all the preceding, instead that it is only the sum of as many preceding as it is necessary to win of matches in succession less one; & by subtracting next that which by this consideration we will have taken too much: I believe you ought be cautioned here in passing that in my Letter of 10 November 1711, I have called p that which I call here $n - p + 1$. Here is, Sir, all that which I have to communicate to you on my method for the pool, I hope that you will be content with it. I have at no point communicated this method to Mr. de Moivre, I believe that if he had seen it he would have recognized that that which he has employed in his Book for the case of three Players, is completely useless for the case of a greater number of Players, & that thus his methods do not always have the advantage to be so general as he thinks. I do not know if Mr. de Moivre has had plan in his Preface[6] to bring so much reproach on you as you believe; for me I hold the methods which you have given in your Book sufficient enough to resolve all the general Problems of Mr. Moivre, most of which differ from yours only in the generality of the algebraic expressions, & I am persuaded that Mr. Moivre himself will do the justice to admit to you that you have pushed this material much further than Mr. Huygens & Mr. Pascal have done, who have given only the first elements of the science of chance, & that after them you have been the first who has published some general methods for this calculus.

[6]From the preface to *De mensura sortis*: "Huygens was the first that I know who presented rules for the solution of this sort of problems, which a French author has very recently well illustrated with various examples; but these distinguished gentlemen do not seem to have employed that simplicity and generality which the nature of the matter demands: moreover, while they take up many unknown quantities, to represent the various conditions of gamesters, they make their calculation too complex; and while they suppose that the skills of gamesters is always equal, they confine this doctrine of games within limits too narrow."

TABLE II

$$\begin{array}{ccccccccccccc}
 & T & + & T & + & V & + & X & + & Y & + & X & +\&c. = 0 \quad \text{No. 1} \\
\text{No. 2} & \| & & \| & & \| & & \| & & \| & & \| & \\
 & A & & A & & B & & \tfrac{1}{2}C + \tfrac{1}{2} \times \overline{B+b} & & \tfrac{1}{4}D + \tfrac{1}{4} \times \overline{C+c} + \tfrac{1}{2} \times \overline{B+2b} & & \tfrac{1}{8}E + \tfrac{1}{8} \times \overline{D+d} + \tfrac{1}{4} \times \overline{C+2c} + \tfrac{1}{2} \times \overline{B+3b} & \&c.
\end{array}$$

Enter

$$\left.\begin{array}{c|c|l}
0 & A & A = \tfrac{1}{2} \times \overline{P-1} + \tfrac{1}{4} \times \overline{P-1+p} + \tfrac{1}{8} \times \overline{P-1+2p} + \tfrac{1}{16} \times \overline{P-1+3p} \cdots \tfrac{1}{2^n} \times \overline{P-1+np-p} + \tfrac{1}{2^n} \times n \\
1 & B & B = \tfrac{1}{2} \times \overline{Q-1} + \tfrac{1}{4} \times \overline{P-1+2p} + \tfrac{1}{8} \times \overline{P-1+3p} + \tfrac{1}{16} \times \overline{P-1+4p} \cdots \tfrac{1}{2^n} \times \overline{P-1+np} + \tfrac{1}{2^n} \times \overline{n+1} \\
2 & C & C = \tfrac{1}{2} \times \overline{R-1} + \tfrac{1}{4} \times \overline{P-1+3p} + \tfrac{1}{8} \times \overline{P-1+4p} + \tfrac{1}{16} \times \overline{P-1+5p} \cdots \tfrac{1}{2^n} \times \overline{P-1+np+p} + \tfrac{1}{2^n} \times \overline{n+2} \\
3 & D & D = \tfrac{1}{2} \times \overline{S-1} + \tfrac{1}{4} \times \overline{P-1+4p} + \tfrac{1}{8} \times \overline{P-1+5p} + \tfrac{1}{16} \times \overline{P-1+6p} \cdots \tfrac{1}{2^n} \times \overline{P-1+np+2p} + \tfrac{1}{2^n} \times \overline{n+3} \\
4 & E & \\
\vdots & \vdots &
\end{array}\right\} \text{No. 3}$$

Exit

$$\left.\begin{array}{cl}
1 \quad P & P = \tfrac{1}{2^{n-1}} \times \overline{V + \overline{n-1} \times v} + \tfrac{1}{2^{n-2}} \times \overline{X + \overline{n-2} \times x} + \tfrac{1}{2^{n-3}} \times \overline{Y + \overline{n-3} \times y} + \tfrac{1}{2^{n-4}} \times \overline{Z + \overline{n-4} \times z} + \cdots \\
2 \quad Q & Q = \tfrac{1}{2^{n-2}} \times \overline{X + \overline{n-1} \times x} + \tfrac{1}{2^{n-3}} \times \overline{Y + \overline{n-2} \times y} + \tfrac{1}{2^{n-4}} \times \overline{Z + \overline{n-3} \times z} + \cdots \\
3 \quad R & R = \tfrac{1}{2^{n-3}} \times \overline{Y + \overline{n-1} \times y} + \tfrac{1}{2^{n-4}} \times \overline{Z + \overline{n-2} \times z} + \cdots \\
4 \quad S & S = \tfrac{1}{2^{n-4}} \times \overline{Z + \overline{n-1} \times z} + \cdots \\
\vdots \quad \vdots &
\end{array}\right\} \text{No. 4}$$

$$\begin{array}{lclclclcl}
B - A & = & \tfrac{1}{2}Q - \tfrac{1}{2}P + \tfrac{1}{4}p + \tfrac{1}{8}p + \tfrac{1}{16}p \cdots \tfrac{1}{2^n} \times p + \tfrac{1}{2^n} & = & \tfrac{1}{2}Q - \tfrac{1}{2}P + t - \tfrac{1}{2}p & = & -\tfrac{1}{2^n} \times V - \tfrac{n}{2^n} \times v + t \cdots & = & V - T \\
C - B & = & \tfrac{1}{2}R - \tfrac{1}{2}Q + \tfrac{1}{4}p + \tfrac{1}{8}p + \tfrac{1}{16}p \cdots \tfrac{1}{2^n} \times p + \tfrac{1}{2^n} & = & \tfrac{1}{2}R - \tfrac{1}{2}Q + t - \tfrac{1}{2}p & = & -\tfrac{1}{2^{n-1}} \times X - \tfrac{n}{2^{n-1}} \times x - \tfrac{1}{2^n} \times v + t \cdots & = & 2X - 2V - v \\
D - C & = & \tfrac{1}{2}S - \tfrac{1}{2}R + \tfrac{1}{4}p + \tfrac{1}{8}p + \tfrac{1}{16}p \cdots \tfrac{1}{2^n} \times p + \tfrac{1}{2^n} & = & \tfrac{1}{2}S - \tfrac{1}{2}R + t - \tfrac{1}{2}p & = & -\tfrac{1}{2^{n-2}} \times Y - \tfrac{n}{2^{n-2}} \times y - \tfrac{1}{2^{n-1}} \times x - \tfrac{1}{2^n} \times v + t \cdots & = & 4Y - 4X - 2x - v \\
 & \text{No. 5} & & \text{No. 7} & & \text{No. 9} & & \text{No. 11} &
\end{array}$$

$$
\begin{array}{lcll}
Q-P & = & -\frac{1}{2^{n-1}} \times \overline{V+\overline{n-1} \times v} + \frac{1}{2^{n-2}} \times x + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots & = -\frac{1}{2^{n-1}} \times V - \frac{n}{2^{n-1}} \times v + p \\
R-Q & = & -\frac{1}{2^{n-2}} \times \overline{X+\overline{n-1} \times x} + \frac{1}{2^{n-3}} \times y + \frac{1}{2^{n-4}} \times z + \cdots & = -\frac{1}{2^{n-2}} \times X - \frac{n}{2^{n-2}} \times x - \frac{1}{2^{n-1}} \times v + p \\
S-R & = & -\frac{1}{2^{n-3}} \times \overline{Y+\overline{n-1} \times y} + \frac{1}{2^{n-4}} \times z + \cdots & = -\frac{1}{2^{n-3}} \times Y - \frac{n}{2^{n-3}} \times y + \frac{1}{2^{n-2}} \times x - \frac{1}{2^{n-1}} \times v + p
\end{array}
$$

No. 6 No. 8

$$
\begin{array}{l|lcll}
A = T & V & = & \frac{T\times 2^n + t\times 2^n - nu}{1+2^n} & = \frac{\overline{T+t}\times 2^n - nu}{1+2^n} \\
B = V & X & = & \frac{V\times 2^n + v\times 2^{n-1} - \frac{1}{2} + t\times 2^{n-1} - nx}{1+2^n} & = \frac{\overline{V+v}\times 2^n - nx}{1+2^n} \\
C = 2X - V - v & Y & = & \frac{X\times 2^n + x\times 2^{n-1} - \frac{1}{2} + v\times 2^{n-2} - \frac{1}{4} + t\times 2^{n-2} - ny}{1+2^n} & = \frac{\overline{X+x}\times 2^n - ny}{1+2^n} \\
D = 4Y - 2X - V - 2x - 2v & & & &
\end{array}
$$

No. 10 No. 12 No. 13.