PROBLEM
ON THE DURATION OF THE GAMES WHICH WE PLAY BY REDUCING.

PIERRE RENARD DE MONTMORT
EXTRACTED FROM
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
2ND EDITION OF 1713, §213–217, PP. 268–277

Pierre & Paul play in a certain number of games by reducing; that is to say, so that Pierre having before him three tokens for example, & Paul three tokens; if Pierre just wins a game, Paul gives to him one of his tokens, leaving thus with two against Pierre four, thus of the rest. We ask how much the odds are that the game which can last to infinity, will be finite by a certain determined number of trials at most.

SOLUTION.

213. Let m be the number of games which are lacking to Pierre or of the tokens which are before Paul; n the number of games which are lacking to Paul or of the tokens which are before Pierre; p the number which expresses in how many trials at most one wishes that the game be finite; a & b the lots of Pierre & Paul. Here is the formula which expresses how much the odds are that Pierre will win the game in p trials at most.
1st series. $1 \times (a^{p-m} + b^{p-m}) + p \times (a^{p-m-1}b + b^{p-m-1}a)$
$+\frac{p.p-1}{1.2} \times (a^{p-m-2}b^2 + b^{p-m-2}a^2) + \frac{p.p-1.p-2}{1.2.3} \times (a^{p-m-3}b^3 + b^{p-m-3}a^3)$
$+$&c. the whole multiplied by $\frac{a^m}{(a+b)^p}$.
2nd series. $-1 \times (a^{p-m-2n} + b^{p-m-2n}) - p \times (a^{p-m-1-2n}b + b^{p-m-1-2n}a)$
$-\frac{p.p-1}{1.2} \times (a^{p-m-2-2n}b^2 + b^{p-m-2-2n}a^2) - \frac{p.p-1.p-2}{1.2.3} \times$
$(a^{p-m-3-2n}b^3 + b^{p-m-3-2n}a^3) -$&c. the whole multiplied by $\frac{a^{m+n}b^n}{(a+b)^p}$.
3rd series. $1 \times (a^{p-3m-2n} + b^{p-3m-2n}) + p \times (a^{p-3m-1-2n}b + b^{p-3m-1-2n}a)$
$+\frac{p.p-1}{1.2} \times (a^{p-3m-2-2n}b^2 + b^{p-3m-2-2n}a^2) + \frac{p.p-1.p-2}{1.2.3} \times$
$(a^{p-3m-3-2n}b^3 + b^{p-3m-3-2n}a^3) +$&c. the whole multiplied by $\frac{a^{2m+n}b^{n+m}}{(a+b)^p}$.
4th series. $-1 \times (a^{p-3m-4n} + b^{p-3m-4n}) - p \times (a^{p-3m-1-4n}b + b^{p-3m-1-4n}a)$
$-\frac{p.p-1}{1.2} \times (a^{p-3m-2-4n}b^2 + b^{p-3m-2-4n}a^2) - \frac{p.p-1.p-2}{1.2.3} \times$
$(a^{p-3m-3-4n}b^3 + b^{p-3m-3-4n}a^3) -$&c. the whole multiplied by $\frac{a^{2m+2n}b^{2n+m}}{(a+b)^p}$.

It is necessary to continue each of these series which are alternatively positives & negatives, to the term where the powers of a & of b are the same in a like term.

In order to demonstrate this formula, I am going to apply it to an example which will serve me to better make understood the reflections which must follow, & to explain myself in a manner more general & more intelligible. Therefore let there be supposed that Pierre has two tokens before him, & Paul three, & that they play by reducing, Pierre wagers that

Date: September 28, 2010.
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he will win the game in fifteen trials at most. If we substitute into the formula above for p, 15; for m, 3; for n, 2; we will find the lot of Pierre

$$\begin{array}{lllllll} = a^{15} & +15a^{14}b & +105a^{13}b^2 & +455a^{12}b^3 & +1365a^{11}b^4 & +3003a^{10}b^5 & +5005a^9b^6 \\ +a^3b^{12} & +15a^4b^{11} & +105a^5b^{10} & +455a^6b^9 & +1365a^7b^8 & +3003a^8b^7 & \\ -1\times a^{13}b^2 & -15a^{12}b^3 & -105a^{11}b^4 & -455a^{10}b^5 & -1365a^9b^6 & & \\ -1\times a^5b^{10} & -15a^6b^9 & -105a^7b^8 & -455a^8b^7 & & & \\ +1\times a^{10}b^5 & +15a^9b^6 & & & & & \\ +1\times a^8b^7 & & & & & & \end{array}$$

So that in the case of $a = b$, we will find that the odds are 12393 against 20375 that Pierre will win in fifteen trials at most. We will find likewise that the odds are 14213 against 18555[1] that Paul will win in fifteen trials at most; so that the odds are 13303 against 3081[2] that the game will be ended & decided in 15 trials at most.

DEMONSTRATION.

1° The first series expresses how many chances in order that Pierre in p trials finds himself at least once to have won a given number m (3) more than Paul: this is evident in regard to all the terms where the exponent of the letter a surpasses the exponent of the letter b by a quantity either $= m$, or greater than m; because we know by *art.* 27, that the formula

$$a^p + pa^{p-1}b + \frac{p.p-1}{1.2}a^{p-2}b^2 + \frac{p.p-1.p-2}{1.2.3}a^{p-3}b^3 + \text{\&c.}$$

expresses all the ways to bring forth either p white faces, or $p-1$ white faces, or $p-2$ white faces, &c. with a number p of dice which have each two faces, one white expressed by a, the other black expressed by b.

It is again true that the other terms of the series, to begin from $a^m b^{p-m}$ (a^3b^{12}) to & including a^9b^6, express all the chances in order that Pierre in p trials finds himself at least once to have won a given number m of games more than Paul: here is the proof of it.

Among all the ways in which we can bring forth a^3b^{12}, that is to say three white faces & twelve black, there is only one way in order that this happen, because it is necessary that the three whites happen the first; & likewise among all the ways in which we can bring forth a^4b^{11}, that is to say four whites & eleven blacks, there are only p (15) ways in order that this happen; because it is necessary that we bring forth all the black faces, excepting one alone after the whites, which can be made in p different ways; since this black face which is not determined to be brought forth after the whites, can be brought forth either in the first, or in the second, or in the third, . . . or in the fifteenth place.

We will find in the same way that it happens so with a^5b^{10}, there are $\frac{p.p-1}{1.2}$ (105) ways, because all the blacks, excepting two, are determined to be brought forth after the whites, & the two faces which are not determined, can be taken two by two in $\frac{p.p-1}{1.2}$ ways, & thus of the other cases, whence it follows that the coefficients of these terms a^p & $a^m b^{p-m}$, $a^{p-1}b$ & $a^{m+1}b^{p-m-1}$, $a^{p-2}b^2$ & $a^{m+2}b^{p-m-2}$, &c. in our example a^{15} & a^3b^{12}, $a^{14}b$ & a^4b^{11}, $a^{13}bb$ & a^5b^{10}, &c. must be the same.

2° It can be made that Pierre does not win the game although it has happened in the course of the game that Pierre has found himself to have won m games more than Paul;

[1] *Translator's note*. Montmort is in error here. It should be 9389 against 6995.

[2] *Translator's note*. Montmort uses the incorrect values indicated in the previous footnote. It should read 31171 against 1597.

because it can happen that he finds himself in this state only after Paul will be seen himself to have won n games more than Pierre.

Our second series expresses how many chances there are not only in order that Pierre win m games more than Paul, but also in order that Paul has won his n games before Pierre has won his m games.

In order that this happen, it is necessary that we bring forth at least n black faces, & at least $n + m$ white faces; because it is necessary that Pierre win n games in order to reduce that which Paul has won before him, & that he win again m games. The other $p - m - 2n$ faces can be either white or black. Now there is only one case for which we bring forth all the other times a white face, & one for which we bring forth a black face; & likewise there are p cases for which we bring forth all the other times less one, a white face, & as many for which we bring forth all the other times less one, a black face; & similarly there are $\frac{p.p-1}{1.2}$ cases for which we bring forth all the other times less two a white face, & as many for which to bring forth a black face, & thus of the rest, &c.

3° We will find likewise the third series expresses how many chances there are that Pierre wins m games; that next Paul reduces these m games, & wins again n games; & that next Pierre reduces the n games won by Paul, & wins again m games.

This series & the others following could be demonstrated in the same manner. Now it is necessary to remark that if we have no regard for the chances that Paul wins n games before Pierre, the first series will give the solution of the Problem, but as Paul can win the n games before Pierre wins the m games, it is necessary to add the second series with the signs of $-$.

Now this series subtracts too much, because it can happen that Pierre wins m games; that next Paul reduces these m games, & wins n games; that Pierre next reduces these n games, & wins anew m games, it is necessary therefore to add the number of cases by which this can happen; & to subtract next the number of chances by which it can happen that Paul wins n games that Pierre reduces these n games, & wins again m games; that next Paul reduces these m games, & wins again n games; & that next Pierre reduces these n games, & wins again m games; & again to add the number of chances by which it can happen that Pierre wins m games, next Paul $m + n$ games, next Pierre $n + m$, next Paul $m + n$, & finally Pierre $n + m$, & again to subtract the number of chances by which Paul wins n games; that next Pierre wins $n + m$ of them, that Paul next wins $m + n$ of them, next Pierre $n + m$, next Paul $m + n$, & finally Pierre $n + m$, & again to add &c. adding & subtracting thus turn by turn as much as the Problem permits it the series which compose our formula.

COROLLARY I.

214. We deduce from the preceding formula the rule which follows.

It is necessary to choose a perpendicular column of which the heading is $p + 2$, *Tab. I, art. I*,[3] & in this column choose the heading which corresponds to the quantity $\frac{p+2-m}{2}$, to add to this number the superiors to the quantity n, take the first 9 terms, for example, if there are lacking nine games to Paul, then by omitting the quantity m, to add the quantity n, to omit the quantity m, & thus in sequence alternately This rule is only an abridgment of the preceding formula: as the statement of it can appear obscure, we are going to clarify it by some examples.

First example. Let it be supposed that Pierre & Paul play each for seven points by reducing, that they are of equal strength, & that we demand how much the odds are that

[3]*Translator's note*. This is Pascal's triangle. See Montmort's version of the table at end.

the game will not endure more than thirty-seven trials, if we substitute into the formula for a and b unity, for p 37, for m, & n, 7, we will have the sum of these terms.

$$\begin{array}{r} \left.\begin{array}{r} 1\\ 37\\ 666\\ 7770\\ 66045\\ 435897\\ 2324784\\ 10295472\\ 38608020\\ 124403620\\ 348330136\\ 854992152\\ 1852482996\\ 3562467300\\ 6107086800 \end{array}\right\}\times 2 \\ 9364199760 \end{array} \qquad \begin{array}{r} \left.\begin{array}{r} -1\\ -37\\ -666\\ -7770\\ -66045\\ -435897\\ -2324784\\ -10295472\\ -38608020 \end{array}\right\}\times 2 \\ -38608020 \end{array} \qquad \begin{array}{r} \left.+1\right\}\times 2 \\ +37 \end{array}$$

the whole divided by 2^{36}.

But if we take care, 1°, that by the formation of the figured numbers the sum of the first f numbers of any perpendicular band g is equal to two times the sum of the first $f-1$ numbers of the $g-1$st perpendicular band, plus the fth term of this $g-1$st perpendicular band. 2° That by following the formula we add & subtract several terms which are the same, which is not useful; we will see that it is much shorter to take consistently to the rule the sum of these terms of the perpendicular band $p+2$,

$$\begin{array}{r} 5471286560\\ 9669554100\\ 5414950296\\ 2707475148\\ 1203322288\\ 472733756\\ \underline{163011640}\\ 38\\ 1 \end{array}$$

of which the sum is 35102333827, that it is necessary to divide by 68719476736, 36th power of 2, in order to have the lot of the one who wagered that the game will not endure more than 37 trials.

Second example. If we suppose $p = 35$ $m = 5$, & $n = 9$, that is to say that Pierre has already won two tokens to Paul, if we wish to know the probability that Paul wins the game in 35 trials at most, we will find following the rule that it is necessary to seek in the 37th perpendicular column of the arithmetic triangle, *art.* I, the $\frac{p+2-n}{2}$ (14th) term, that to this term it is necessary to add the four following, these five terms are

$$\begin{array}{r}2310789600\\1251677700\\600805296\\254186856\\94143280\end{array}$$

then to omit the quantity n of them; that is to say the 9 following; then to add m of them, that is to say the 5 following which are found to be all zero. So that the lot of the one who would wager that Paul would win the game in 35 trials at most, would be 4511602732 divided by the 35th power of 2.

It is apropos to observe that we can again abridge the calculation, & to have in this case, for example, only three terms in place of the five above,

$$\begin{array}{r}3562467300\\854992152\\94143280\end{array}$$

by serving oneself the terms of the perpendicular band $p + 3$ interposed two by two.

Mr. (Nicolas) Bernoulli who I have invited to work on this problem which has seemed to me quite difficult, sent to me a very lovely & very general solution of it in his letter of 26 February 1711; & as the manner in which it is stated had appeared to me a little obscure, he had the willingness to explain it to me quite at length in his Letter of 10 November of the same year; I noticed then, & the Reader will note it easily, that we both have followed the same route.

I was surprised in reading the past year the Treatise of M. Moivre *De Mensura Sortis*, to encounter a different solution of this Problem. The Author seeks how many chances there are in order that the game not finish. It is necessary to confess that the idea which he has followed is quite simple, & that it is very ingenious; however it is certain that ours is of simpler usage, & that the calculation of it is incomparably shorter, especially if we wish to employ logarithms, which would seem impossible in the method of M. Moivre.

I have sought, by availing myself of logarithms, what was the lot of the one who would wager that m & n being $= 9$, the game will end in 61 games, & I have found in less than one hour this great number 581928000 000 000 000 divided by 2^{60}.[4] By the method of M. Moivre it is necessary to raise $1 + 1$ to the 9th power, & to make 26 multiplications of this number by $1 + 2 + 1$, subtracting the last two terms at each operation, which seems to me to demand an immense calculation.

COROLLARY II.

215. When the Players each have only two tokens before them, the solution of this Problem is contained in this geometric progression $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$+&c. So that the sum of the first two expresses how much the odds are that the game will be decided in four games at most, & the sum of the first three expresses how much the odds are that it will be decided in six trials at most, &c.

And when each of the Players has three tokens before him, we find in this formula $\frac{1}{4} + \frac{3^1}{4^2} + \frac{3^2}{4^3} + \frac{3^3}{4^4} + \frac{3^4}{4^5}$+&c. the reason of the advantage or of the disadvantage that the odds are that the game will be decided in any certain number of games. The first term of this series expresses the lot of the one who would wager that the game will be decided in

[4]*Translator's note*. The exact value is 581896325969661088.

three trials. The sum of the first two terms expresses the lot of a Player who would wager that the game will be decided in five trials. The sum of the first three terms expresses the lot of a Player who would wager that the game will be decided in seven trials, &c. But when the number of tokens that each of the Players has before him, is greater than three: we no longer find the convenience of geometric progressions, & it is necessary to have recourse to the preceding solution.

COROLLARY III.

216. Let p be the number which expresses what odd natural number of tokens which each of the Players has before him. I have found that the formula $3pp-3p+1$ expresses in how many trials at most we can wager with advantage that the game will be finite. So that, for example, each player having nineteen tokens before him; by substituting in the formula for p, 10, since the tenth odd natural number is nineteen, we find that we can wager with advantage that the game will not endure more than 271 trials. I have sought a parallel formula for the case where the number of games which are lacking to each of the players is even, but I have not been able to find it.

COROLLARY IV.

217. Here is a Table which I amused myself by making some time ago in just playing a game of Piquet, it was to the full hundred; in order to win it was necessary to have six games, & we played always by reducing in the manner that we have explained. It will happen that the Players after having played thirty or forty games, one or the other approaching from time to time to the goal without being able to attain it, took the decision to renounce it, & to separate themselves without ending it; but as one of the Players had three points or three marked games before him, we agreed that it was correct to leave the money of the game (it was eight Louis which made the 128 liv.) most equally as he himself could; I found that the Player who had three tokens of the other Player, must withdraw 96 liv. & the other 32 liv.

TABLE.

Pierre has 1 of the tokens of Paul, his chance is 7 against 5.
Pierre has 2 of the tokens of Paul, his chance is 2 against 1.
Pierre has 3 of the tokens of Paul, his chance is 3 against 1.
Pierre has 4 of the tokens of Paul, his chance is 5 against 1.
Pierre has 5 of the tokens of Paul, his chance is 11 against 1.

The illustrious Mr. (Jean) Bernoulli[5] noticed that the lots followed this order.

$$n+0 \text{ against } n-0$$
$$n+1 \text{ against } n-1$$
$$n+2 \text{ against } n-2$$
$$n+3 \text{ against } n-3$$
&c.

Mr. (Nicolas) Bernoulli has also deduced from his general formula this Problem & many others. See his Letter of 26 February 1711.

[5] See his Letter of 17 March 1710.

Extract of a Letter of M. (Jean) Bernoulli to M. de Montmort (p. 295–296)

At Basel this 17 March 1710.

Page 178, line 9. Here is a Table. I myself marvel that you have not observed the uniformity of this Table, & the great facility with which we construct a general Table for some number of games that we always play by reducing; because if the number of games in which Pierre & Paul agree to play n: here is the Table of their lots.

TABLE.

If Pierre has no point,	his lot is	$n+0$	against	$n-0$
one point,	$\cdots$	$n+1$	$\cdots$	$n-1$
two points,	$\cdots$	$n+2$	$\cdots$	$n-2$
three points,	$\cdots$	$n+3$	$\cdots$	$n-3$
four points,	$\cdots$	$n+4$	$\cdots$	$n-4$
$\vdots$		$\vdots$		$\vdots$
n points,	$\cdots$	$n+n$	$\cdots$	$n-n$

You see that their lots go in arithmetic progression, the one ascending, the other descending: in fact your Table that you have made for the number of six only, agrees with my general one; because the lot

of 7	against	5	is the same thing as	$6+1$	against	$6-1$
2		1	$\cdots$	$6+2$	$\cdots$	$6-2$
3		1	$\cdots$	$6+3$	$\cdots$	$6-3$
5		1	$\cdots$	$6+4$	$\cdots$	$6-4$
11		1	$\cdots$	$6+5$	$\cdots$	$6-5$

Extract of a Letter of M. de Montmort to M. Jean Bernoulli

At Montmort 15 November 1710 (pg. 306–307)

Page 178.[6] This is a Problem which could excite your curiosity, the one where the question is to determine how long the trials must endure when one plays by reducing. I have given in the Corollary, *page 184*[7] the solution of the case where one would play to three matches of Piquet: I have the general solution of this Problem. I would send it to you if I believed that it would give you pleasure. Sir, your nephew, who appears capable to me by his ability with the most difficult things, & who besides is young & has perhaps the leisure, ought to seek the solution which is assuredly worthy of you. I have found almost without calculation that the lot of the one who would wager that the game will not endure more than 26 trials, will be $\frac{16607955}{33554432}$, what which shows that there would be little disadvantage, & that the one who would wager that the game will not endure more than 28 trials, will be $\frac{70970250}{133432831}$, which being greater than $\frac{1}{2}$, shows that in this case there would be advantage. I believe also to have found that in twelve games there would be the advantage to wager that the game will end in less than 124, & that there is disadvantage in 122.

[6]See page 268.
[7]See page 276.

Extract of a Letter of M. (Nicolas) Bernoulli to M. de Montmort (p.309–311)

At Basel 26 February 1711.

I have found the same although under another expression by the method of combinations. The Problem that you propose on the game which is played in many games by reducing is quite difficult; nonetheless seeing that you wished that I find a solution of it, I have applied myself, & I have found a general rule in order to express the lot of the one who would wager that one of the Players will have won in such number of trials as we will wish, be that they play in one equal or unequal game, be that one has already won some games or none: here it is in words. Let the two Players be Pierre & Paul, the number of parts which are lacking to Pierre$= m$, the number of parts which are lacking to Paul$= n$, their sum$= m+n=s$, the number of cases favorable to Pierre$= p$, the number of cases favorable to Paul$= q$, their sum $= p+q=r$, the number of trials $= h = m+2k$, the number of times that s is contained in $k=t$; this put, I say that the difference between the sum of all the possible values (that is to say by putting for t all the values which it can have from 0 to the greatest) of this series

$$\begin{aligned}1\times&(p^{2k-2ts}+q^{2k-2ts})+h\times(p^{2k-2ts-1}q+q^{2k-2ts-1}p)\\&+\frac{h.h-1}{1.2}\times(p^{2k-2ts-2}qq+q^{2k-2ts-2}pp)\\&+\frac{h.h-1.h-2}{1.2.3}\times(p^{2k-2ts-3}q^3+q^{2k-2ts-3}p^3)\\&+\text{\&c. to }\frac{h.h-1.h-2\cdots h-k+ts+1}{1.2.3.4\cdots k-ts}\times pq^{k-ts}\end{aligned}$$

the whole multiplied by $\frac{p^{ts+m}q^{ts}}{r^h}$, & the sum of all these values of this here

$$\begin{aligned}1\times&(p^{2k-2ts-2n}+q^{2k-2ts-2n})\\&+h\times(p^{2k-2ts-2n-1}q+q^{2k-2ts-2n-1}p)\\&+\frac{h.h-1}{1.2}\times(p^{2k-2ts-2n-2}qq+q^{2k-2ts-2n-2}pp)\\&+\text{\&c. to }\frac{h.h-1.h-2\cdots h-k+ts++n+1}{1.2.3.4\cdots k-ts-n}\times pq^{k-ts-n},\end{aligned}$$

the whole multiplied by $\frac{p^{ts+s}q^{ts+n}}{r^h}$, will express the lot of the one who would wager that Pierre will win the game in at least h trials. If k is smaller than $ts+n$; that is to say, if after having divided k by s, the rest of the division is smaller than n, it is not necessary in the last series to put for t all the values from 0 to t, but only to $t-1$. In order to have the lot of the one who would wager that Paul will win it in h trials, it will be necessary only to substitute into this formula the letters q, p, n, m, in place of p, q, m, n. The sum of these two lots together will be the lot of the one who would wager that the game will be decided in h trials. The application of this formula to some particular cases, when $p=q=1$, is quite easy; I have found not more than you, since without calculation, that for six games the lot of the one who would wager that the game will be ended in 26 trials will be $\frac{16607955}{33554432}$, & in 28 trials $\frac{35485125}{67108864}$, or $\frac{7090250}{134217738}$, & not $\frac{70970250}{133432831}$, as you have written in error; but for twelve games I have found that we can already wager with advantage that the game will be ended in 110, & it would be disadvantageous to wager that it will end in 108 trials; because the lot for these two numbers of trials will be $\frac{329\cdots}{649\cdots}$ & $\frac{810\cdots}{1622\cdots}$,[8] it must be therefore that you yourself are mistaken, since you say that we can wager with advantage when the game will be decided only in 124 trials. It must be however to confess that it is necessary by groping in order to find when the chance will be $\frac{1}{2}$; this is why if you have a better method than that here, I pray you to communicate it to me, & I will be much obliged to you. It is clear that this formula, which I just gave, will serve also to find the lot of the same Players; because for this end it will be necessary only to suppose that the number of trials is infinite, by putting therefore $h,\ k,$ & $l =$*inf.* we will find that the lot of Pierre will be

$$= \frac{(p+q)^h \times p^s - p^{s-n}q^n}{r^h \times p^s - q^s} = \frac{p^s - p^m q^n}{p^s - q^s};$$

& consequently that of Paul $\frac{p^m q^n - q^s}{p^s - q^s}$, which I have found formerly by a different way from that which I have followed in the research on this Problem. If $m = n$, & $s = 2m$, their lots will be as $p^{2m} - p^m q^m$ & $p^m q^m - q^{2m}$, or as p^m & q^m; & by supposing $m = 12$, $p = 9$, $q = 5$, we will have 9^{12} & 5^{12} for the chances of Pierre & Paul, which is the case of the fifth Problem of Huygens. If $p = q$, the lots of the two Players are as n & m, which is found easily by dividing $p^s - p^m q^n$, & $p^m q^n - q^s$ by $p - q$; because we will have by this division two geometric progressions, of which the number of terms of the 1st will be $= n$, & that of the 2nd $= m$, & of which the terms, by supposing $p = q$, will become all equal. If $p = q$, & $s = m + m = 12$, we have the case of page 178[9] of your Book.

Extract of a Letter of M. de Montmort to M. (Nicolas) Bernoulli (pag. 315–316)

To Montmort the 10 April 1711.

I admire your formula for the duration of the games that we play by reducing; I sense that it is quite correct, but I am forced to say to you that I do not understand it. You have given me great pleasure, for me to facilitate the understanding of it, by making application of it in an example: for example, in the one where we play to six games, & where we find that there is advantage to wager that it will endure less than 28 trials. It is true that I deceived myself in the denominator, you have also deceived yourself inadvertently, it must be 134217728, & not 13421738: these kinds of errors slip in quite easily, when we are tired from a long calculation. I begin to doubt in any case as you that we can wager with advantage that playing to eleven games the game will end in 124 games, & not in 122. I have made that which I have been able to recall my ideas on this Problem which is assuredly quite difficult & quite abstract. I have not been able to find the papers where the demonstrations of these Problems are figured, & I believe that they are in Paris: immediately as I will be there I will do for you part of that which I have found on this matter. I will say to you only that we both have followed a quite different path, which you will understand quite easily, Sir, when you know that this number 70970250 is the sum of these six 34597290, 20030010, 10015005, 4292145, 1560780, 475020, which are the 7, the 8, the 9, the 10, the 11 & the 12th terms of the 30th perpendicular band. I find in a Book where I have put formerly some remarks that the odds are $\frac{35103333817}{2\times 24359738368}$ that playing

[8] *Translator's note*. These values are correct. Indeed, the exact probability that the game terminate in 110 trials is $\frac{329756296122611431546168042626736}{649037107316853453566312041152512}$ and the exact probability that the game terminate in 108 trials is $\frac{81057262276448668848223046732461}{162259276829213363391578010288128}$. The first quotient is approximately 0.5080700200 and the second is 0.4995539476.

[9] *Translator's note*. This refers to the first edition.

to 7 games the game will be ended in 37 games at least,[10] & $\frac{8338160273}{2\times 8589934592}$ that it will be ended in less than 35,[11] which shows that there is advantage in 37, & disadvantage in 35. You can verify by your formula if this calculation is correct: besides your formula amazes me for its generality; I see that you draw from it the best as can be the fifth Problem of M. Huygens, & that of page 178.

Extract of a Letter of M. (Nicolas) Bernoulli to M. de Montmort (p. 324–326)

At Basel 10 November 1711.

I am displeased that the series

$$\begin{aligned}1\times&(p^{2k-2ts}+q^{2k-2ts})\\&+h\times(p^{2k-2ts-1}q+q^{2k-2ts-1}p)\\&+\frac{h.h-1}{1.2}\times(p^{2k-2ts-2}qq+q^{2k-2ts-2}pp)\\&+\frac{h.h-1.h-2}{1.2.3}\times(p^{2k-2ts-3}q^3+q^{2k-2ts-3}p^3)+\&\text{c.}\\ \text{to }&\frac{h.h-1.h-2\cdots h-k+ts+1}{1.2.3.4\cdots k-ts}\times pq^{k-ts}\times\frac{p^{ts+m}q^{ts}}{r^h},\end{aligned}$$

&

$$\begin{aligned}1\times&(p^{2k-2ts-2n}+q^{2k-2ts-2n})\\&+h\times(p^{2k-2ts-2n-1}q+q^{2k-2ts-2n-1}p)\\&+\frac{h.h-1}{1.2}\times(p^{2k-2ts-2n-2}qq+q^{2k-2ts-2n-2}pp)+\&\text{c.}\\ \text{to }&\frac{h.h-1.h-2\cdots h-k+ts++n+1}{1.2.3.4\cdots k-ts-n}\times pq^{k-ts-n}\times\frac{p^{ts+s}q^{ts+n}}{r^h}\end{aligned}$$

that I have given in order to determine the duration of the games that we play by reducing, has not been sufficiently intelligible to you. It is in these sorts of matters sometimes difficult to make them well understood, especially when we do not take care to avoid all the ambiguities that can be encountered, as I believe that it has happened to me; because it seems to me that the cause for what you have not understood by me, consists only in that which I have said, that it is necessary to put for t all the values that it can have from 0 to the greatest, in which there is a little ambiguity that I could have avoided by putting in the formula for t one letter, for example v, & by saying that in the application it is necessary to put for v successively $0,\ 1,\ 2,\ 3,\ 4,$ &c. to t, which expresses the number of times that s is contained in k. In order to give it to you in greater clarity, I am going to demonstrate how I have deduced from these series a general rule for the games which are played in an equal game, to which I will apply next in some particular cases of seven games. It is clear that when $p=q=1$, & $r=p+q=2$, these two series are changed into this one here,

$$1\times 2+h\times 2+\frac{h.h-1}{1.2}\times 2+\frac{h.h-1.h-2}{1.2.3}\times 2+\&\text{c.}$$

[10]*Translator's note*. The text says "au moins" but the probability given is that the contest terminate in at most 37 games. The value is also incorrect and should be $\frac{35102333827}{68719476736}$ which is approximately 0.5108.

[11]*Translator's note*. The text says "en moins de." However, this makes no sense in light of the fact that the probability the contest terminates in 35 games is actually $\frac{8338160273}{17179869184}$, which is approximately 0.4853.

to $\frac{h.h-1.h-2\cdots h-k+vs+1}{1.2.3\cdots k-vs} \times 1$, the whole divided by 2^h, & $1 \times 2 + h \times 2 + \frac{h.h-1}{1.2} \times 2 + \frac{h.h-1.h-2}{1.2.3} \times 2$+&c. to $\frac{h.h-1.h-2\cdots h-k+vs+n+1}{1.2.3\cdots k-vs-n} \times 1$. The whole divided by 2^h (I put here v in place of t, for the reason that I just said.) Now the terms of these series are nothing but the those of the perpendicular band of the arithmetic triangle of M. Pascal, of which the heading is expressed by $h+1$, each multiplied by 2, except the last; whence it follows that their sums are correctly the sums of as many of the terms of the following band, of which the heading is $h+2$; since therefore the number of these terms of the first series is $k - vs + 1$, & that of the second $k - vs - n + 1$, the sums of all the possible values of these two series, by taking for v successively 0, 1, $2, 3$, 4, &c. to t, will be $\boxed{k+1} + \boxed{k-s+1} + \boxed{k-2s+1} + \boxed{k-3s+1}$ +&c. And $\boxed{k-n+1} + \boxed{k-s-n+1} + \boxed{k-2s-n+1} + \boxed{k-3s-n+1}$ +&c. By this arbitrary mark $\boxed{k+1}$ I intend the sum of as many of the first terms of the perpendicular band which correspond to the heading $h+2$, as there are units in $k+1$. The difference of these two sums $\boxed{k+1} - \boxed{k-n+1} + \boxed{k-s+1} - \boxed{k-s-n+1} + \boxed{k-2s+1} - \boxed{k-2s-n+1}$ +&c. divided by 2^h will express the lot of the one who would wager that Pierre will win the game in less than h trials. In order to apply this to some particular cases; we suppose, for example, that we play for seven games, & that we wish to know how much we could wager at the beginning of the game that one of the Players, for example Pierre, will win the game in less than 35 trials. We will have $m = n = 7$, $s = m + n = 14$, $h = 35 = 7 + 2k$: therefore k will be $= 14$, & $t = 1$; & the formula $\boxed{k+1} - \boxed{k-n+1} + \boxed{k-s+1}$ −&c. divided by 2^h will be changed into this one here $\boxed{15} - \boxed{8} + \boxed{1}$ divided by 2^{35}, which indicates that it is necessary to divide the sum of the 15, 14, 13, 12, 11, 10, 9th & 1st term of the 37th perpendicular band, that is to say 8338160273, by 2^{35} in order to have the lot of the one who would wager that Pierre will win in less than 35 games, & that it is necessary to divide it by 2^{34} in order to have the lot of the one who would wager that the game will be ended in 35 games, conforming to our calculation; but for 37 games I find that it must be $\frac{35102333827}{2\times 34359738368}$, & not $\frac{35103333817}{2\times 34359738368}$, as you have written by error. If we suppose that $m = 5$, $n = 9$, $s = 14$, & $h = 35$, that is to say, that Pierre has already won two games, & that we wish to know the probability that Pierre or Paul will win the game in 35 trials, we will find $\boxed{16} - \boxed{7} + \boxed{2}$ divided by 2^{35} or $\frac{13914410549}{34359738368}$ for the lot of the one who would wager that Pierre will win the game in 35 trials, & $\boxed{14} - \boxed{9}$ divided by 2^{35} or $\frac{4511602732}{34359738368}$ for the lot of the one who would wager that Paul will win it in 35 trials. The sum of these two lots $\frac{18426013281}{34359738368}$ will express the lot of the one who would wager that the game will be decided in 35 trials, which shows that it would be to the advantage. I believe that this will suffice for you to make understood the sense of my formula: we pass to other things.

Extract of a Letter of M. de Montmort to M. (Nicolas) Bernoulli (p. 344–345)

At Paris the 1 March 1712.

I understand perfectly your formula for the games by reducing; it assuredly needed explication in order to be understood; I have seen with surprise & admiration that it was not much different from mine. You yourself will have without doubt noticed in my last Letter, when I had sent you that this number 70970250 is the sum of the six 34597290, 20030010, 10015005, 4292145, 1560780, 475020, which are the 7, the 8, the

556790260
3796297200
2310789600
1251677700
600805296
254186856
34143280
30260340
8347670

9, the 10, the 11 & the 12th terms of the 30th perpendicular band. I do not know if it there must be a 3 in place of a 4 in this numerator 13914410549, I find 13914410539: this number is formed from the sum of these nine, & again from these two 36. 1.

It would not be useful that I report to you here my method all at length, it is slightly different from yours only in the manner of statement, with the exception that I have had in view only the supposition of equal chances for one & the other Player, instead as you suppose them in any ratio, here it is briefly. Let p be the number of trials, m the number of games which are lacking to Pierre, n the number of games which are lacking to Paul, it is necessary to choose a perpendicular column of which the heading is $p+2$, & in this column to choose the heading which corresponds to the quantity $\frac{p+2-m}{2}$; to add to this number the superiors to the quantity n. Take the nine first terms, for example, if there is lacking nine games to Paul, then by omitting the quantity m, to add the quantity n, to omit the quantity m, & thus in sequence alternately, to divide these numbers by 2^p; we will have a fraction which expresses how much the odds are that Pierre will win the game at least by as many trials as p expresses units. If we wish to have for Paul that which we have for Pierre, it is necessary to put everywhere m in place of n, & n in place of m.

Extract of a Letter of M. de Montmort to M. (Nicolas) Bernoulli (p. 368–369)

At Montmort this 5 September 1712.

The rest of the Book[12] contains seven propositions on an extremely curious matter to which I have the first thought; namely how long a game must endure where we always play by reducing, that which I explain in my Book page 178, and better again in my last Corollary page 184 where I give this series $\frac{1}{4}+\frac{3^1}{4^2}+\frac{3^2}{4^3}+\frac{3^3}{4^4}+\frac{3^4}{4^5}+$&c. in order to determine how much the odds are that the game will be ended in less than $3,\ 5,\ 7,\ 9,\ 11,\ 13,$&c. games to infinity. I end this Corollary with these words: *we will find without much pain some similar formulas for the other cases, & the search for them will seem curious.* The truth is however that this Problem is not at all easy, even with the help of the particular formula for the case of three games. I see with pleasure that M. Moivre came to the end of this Problem in total, & that his solution agrees perfectly with ours. I am quite at pain to know how this learned Geometer arrived at his method to raise $a+b$ to the power n, to subtract the extreme terms of this product, & to multiply again the result by the square of $a+b$, & thus in sequence as many times as there are units in $\frac{1}{2}$d. a solution of this nature surprises me, & so much the more that the Author who had supposed equal the number of chances for Pierre & for Paul coming to suppose them in any ratio, is obliged to take

[12]*Translator's note*. This refers to Moivre's *De Mensura Sortis*.

another route; instead of that by you & by me the method is the same for the general & particular solution; this does not prohibit that I do not esteem this discovery, & in general all his Work, to which I compliment myself to have given occasion, by opening the course first. It has appeared to me first most singular that he has filled it with the same things of which we ourselves conversed in our Letters; but it is natural that having made his work on mine, & wishing to push these matters, the same ideas came to him as to us. I would only wish, & it seems to me that equity would demand it, that he had recognized with frankness that which I have right to claim in his Work. I am obliged to him of some very honest expressions of which he avails himself in his Preface in speaking of me & of my Book; but I know not in truth on what it is founded when he says. . .I cannot fathom for what reasons this Author makes these reproaches to me, & what motive carries him to pronounce against me, by leaving me only the merit of having applied to some examples the supposed rules of M. Huygens, I call on this judgment to the Geometers who will read this that M. Huygens & M. Pascal, of whom the Author does not speak, have given on this matter.

Extract of a Letter of M. (Nicolas) Bernoulli to M. de Montmort (p. 375)

At London this 11 October 1712.

I have the pleasure to see here often M. de Moivre who has made me a present of his Book *De Mensura Sortis*. He has said to me that he has sent an Exemplary of it to you also, & he waits with impatience your sentiment on this Work. You will be astonished to find many Problems that we have solved, & among others also the one of the duration of the games by reducing, which he has resolved in a manner, although different from ours, nonetheless very lovely & very curious.

Extract of a Letter of M. (Nicolas) Bernoulli to M. de Montmort (p. 380)

At Brussels this 30 December 1712.

The method of M. de Moivre for the duration of the games that we play by reducing is very natural & founded on this that it is always necessary to subtract the cases for which it can happen that one of the Players win the écus of the other; the method when the number of écus of both is equal, is not different from what he uses when the number of their écus is unequal, what one makes in the first two operations all at once, because of the equality that there is on all sides.

Table of M. Pascal for the combinations.(pg. 2)

1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13
		1	3	6	10	15	21	28	36	45	55	66	78
			1	4	10	20	35	56	84	120	165	220	286
				1	5	15	35	70	126	210	330	495	715
					1	6	21	56	126	252	462	792	1287
						1	7	28	84	210	462	924	1716
							1	8	36	120	330	792	1716
								1	9	45	165	495	1287
									1	10	55	220	715
										1	11	66	286
											1	12	78
												1	13
													1