PROBLEM
ON
THE GAME OF BASSETTE

PIERRE RENARD DE MONTMORT
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
2ND EDITION OF 1713, §115–124, PP. 144–156

EXPLICATION OF THE RULES.

115. In this game, as in that of Pharaon, the Banker holds an entire deck composed of fifty-two cards. After he has shuffled them, & after each Player or Punter has put a certain sum on a card taken at will, the Banker turns the deck, putting the bottom above; so that he sees the bottom card. Next he draws all the cards two by two until the end of the game, by commencing with the second. Here are the other rules of the game.

1°. The first card is for the Banker; but he takes only the two-thirds of the stake of the Punter when he brings forth his card, & this is called *facing*. The second is entirely for the Punter, the third entirely for the Banker, & thus in sequence alternately. It is necessary to remark that when a card has won or lost it no longer appears in the game, at least one does not replace it anew. Thus, for example, the card of the Punter being a King, if the first card of the deck is a Queen, the second a King & the third also a King, the Banker who says in drawing the cards, *King has won, King has lost* (this is understood of the Punter) will lose the stake of the Punter, although naturally the second King had made it winning, if the first card of the deal had been no King at all.

2°. When the Punters wish to take a card in the course of the game, it is necessary that the deal be low, that is to say that the Banker drawing them, as I have said two by two, has put the last deal or pair of cards on the mat, so that the card which remains revealed is losing for the Punters. Then if a Punter takes a card, the first card which the Banker will draw will be null in regard to this Punter, although it is favorable to the other Players; if it comes second, it will be faced, that is to say that the Banker will take $\frac{2}{3}$ of that which this Punter will have put on the card: if it comes in sequence, it will be in pure gain or in pure loss for the Banker, according as it will come, either first, or second from a deal.

3°. The last card, which must be for the Punter, is null.

PROPOSITION IX.
FIRST METHOD.
FIRST CASE.

We suppose that the Banker having six cards in his hands, the Punter takes one of them which is one time in these six cards, that is to say in the five covered cards. We ask what is the lot of the Banker with respect to this card of the Punter. For example, if the Punter

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puts an écu on his card, we ask to what part of the écu can be evaluated the advantage of the Banker.

116. Let the sought lot be expressed by S, & the stake of Paul by A.

If we imagine the one hundred-twenty different arrangements that five cards expressed by the letters a, b, c, d, f can receive, put under five columns, each of twenty-four arrangements; we will remark, 1°, that that where the letter a occupies the first place, gives A to the Banker. 2°. That in each of the four other columns, the letter a is found six times in the 2nd place, six times in the 3rd place, six times in the 4th, & six times in the fifth; whence it follows that we will have

$$S = \frac{24 \times A + 4 \times \overline{6 \times \frac{5}{3}A + 6 \times 2A + 6 \times 0 + 6 \times A}}{120} = \frac{136}{120}A = A + \frac{2}{15}A;$$

& consequently if A designates an écu worth sixty sols, Paul taking a card, under the conditions of the present Problem, would make to Pierre the same advantage as if he would give to him eight sols in pure gift.

We can next consider the other thing, by taking care that of these five columns, the first will give $24A$, the second $24 \times \frac{5}{3}A$, the third 24×0, the fourth $24 \times 2A$, & the fifth $24A$.

If the card which the Punter takes is only one time among the covered cards of the Banker of which the number is expressed by p, we will have

$$S = \frac{3Ap + 2A}{3p}.$$

SECOND CASE.

We suppose that the Banker holding six cards, the Punter takes one of them. Now as the card of the Punter is found either two times, or three times, or four times in these six cards, & as this varies the advantage of the Banker, it is à propos to seek what is his lot in all the variations of this second case. I will begin by examining what is his lot under the supposition that the card of the Punter is two times in the hand of the Banker.

117. Let the five covered cards of the Banker be designated by the letters a, b, c, d, f, of which any two, for example, a & f, express that of the Punter. We will remark, 1°, that the one hundred-twenty different possible arrangements that the five cards can receive, being put under five columns each of twenty-four arrangements, of which the first begins with a, the second with b, the third with c, &c. the two columns which begin with a & with f give A to the Banker, because they are indifferent for the Banker & for the Punter. 2°. That each of the three other columns contain twelve arrangements which give to the Banker $\frac{5}{3}A$, these are those where a & f are in the second place; & four arrangements which give $2A$ to the Banker, that is to say which make him win. This will be discovered easily from the Table here joined which represents the second column, which is that where

b holds the first place.

$bacdf$	$bcadf$	$bdacf$	$bfadc$
$bacfd$	$bcafd$	$bdafc$	$bfacd$
$badcf$	$bcdaf$	$bdcaf$	$bfcad$
$badfc$	$bcdfa$	$bdcfa$	$bfcda$
$bafcd$	$bcfad$	$bdfac$	$bfdac$
$bafdc$	$bcfda$	$bdfca$	$bfdca$

It is clear that the first & the last of these four columns give $\frac{5}{3}A$ to the Banker, & that each of the two others contain two arrangements which give $2A$ to the Banker; these are these here, $bcdaf$, $bcdfa$, $bdcaf$, $bdcfa$. We will have therefore

$$S = \frac{2 \times 24A + 3 \times \overline{2 \times 6 \times \frac{5}{3}A + 2 \times 2 \times 2A}}{120} = \frac{11}{10}A = A + \frac{1}{10}A.$$

2°. In order to find what is the lot of the Banker when the card which the Punter takes is three times in the five cards of the Banker. We will observe that of the aforesaid five columns there are three which give A to the Banker, & two which contain each eighteen arrangements which give $\frac{5}{3}A$ to the Banker. This has no need of proof. We will have therefore

$$S = \frac{3 \times 24A + 2 \times 18 \times \frac{5}{3}A}{120} = \frac{132}{120}A = A + \frac{1}{10}A.$$

3°. In order to find what is the lot of the Banker when the card which the Punter takes is four times in the five cards covered by the Banker. We will observe that of the aforesaid five columns there are four which give A to the Banker, & one which gives to him $\frac{5}{3}A$. We will have therefore

$$S = \frac{4 \times 24A + 24 \times \frac{5}{3}A}{120} = A + \frac{2}{15}A.$$

THIRD CASE.

We suppose that the deck being composed of eight cards, of which the first is uncovered, the Punter takes one of them which is two times in these eight cards. We ask what is the lot of the Banker with respect to that card.

118. Let the seven covered cards be expressed by the seven letters a, b, c, d, f, g, h, of which two, namely a & f, designate that of the Punter. Let also, as above, S be the sought lot, & A the stake of Paul. This put,

We will observe, 1°, that putting the five thousand forty different arrangements that the seven letters can receive on seven columns each of seven hundred twenty arrangements, the column which begins with a & that which begins with f, each will give A to the Banker. 2°. That if we imagine each of the five others partitioned anew into six others of one hundred twenty arrangements each, the two from among these six where a & f occupy the second place, will give $\frac{5}{3}A$ to the Banker. 3°. That the four other columns from among these six have each forty-eight arrangements which give $2A$ to the Banker. In order to see easily it is necessary to suppose that one of the five columns subdivided into six others, is that which begins with b, & to consult the Table which has served in the solution of the preceding case. We will remark first that the first and the last column of this Table being varied as much as it is possible with the two new letters g & h, a remaining in the second place, they will furnish each one hundred twenty arrangements which give $\frac{5}{3}A$ to the Banker. In regard to the four other columns of one hundred twenty arrangements each, in which the letters c, d, g, h would occupy the second place after b, it is easy to see that it suffices to examine one of them, since all four give the same lot to the Banker. Let the third column of the Table be that which we wish to examine. It is necessary to take care that each of the four arrangements $bcadf$, $bcafd$, $bcfad$, $bcfda$ being varied with the two new letters g & h, as much as it is possible, in such a way nonetheless that c remains in the second place, that is to say immediately after b, gives six new arrangements which are winning to the Banker, and give to him $2A$. For example, $bcadf$ furnishes these here,

$$\begin{matrix} bcgadfh & bchadfg \\ bcgadhf & bchagfd \\ bcgahdf & bchafgd \end{matrix}$$

It is thus of the three others of them, since g being in front of a or f, it can be found in three different places; & since h being in front of a or f, it can be found in three different places, a or f remaining always in the fourth.

We will find likewise that the two arrangements $bcdaf$, $bcdfa$ being varied as much as it is possible with g & h, in such a way nonetheless that c is always in the second place, furnishes each twelve arrangements which give $2A$ to the Banker; because in $bcdaf$, g & h can be arranged in six ways with d, & in six different ways with f, a remaining in the fourth place; & likewise in $bcdfa$, g & h can be arranged in six ways with d, & in six different ways with a, f remaining always in the fourth place. From all this it follows that we will have

$$S = \frac{2 \times 720A + 5 \times \overline{2 \times 120 \times \frac{5}{3}A + 4 \times 48 \times 2A}}{5040} = \frac{536}{504}A = A + \frac{4}{63}A.$$

2°. In order to find what is the lot of the Banker when the card which the Punter takes is three times in the seven covered cards of the Banker.

Let the seven cards of the Banker be expressed as above by the letters a, b, c, d, f, g, h, of which any three, for example a, d, f, designate the card of the Punter. This put,

We will observe, 1°, that putting the five thousand forty different arrangements that the seven cards can receive on seven columns of seven hundred twenty arrangements each, the three which begin with the letters a, d, f give A to the Banker, that which is evident. 2°. That distributing each of the four others into seven columns of one hundred twenty arrangements each, the three columns from among these six where the letters a, d, f will take the second place, gives $\frac{5}{3}A$ to the Banker. 3°. That each of the three other columns will contain thirty-six arrangements which will give $2A$ to the Banker. In order to be assured of this, we can consult the Table of *art. 117*, & remark that each of the arrangements of the second column of the Table where b is in the first place, & c in the second, can by the mixing of the two letters g & h, receive only six arrangements which give $2A$ to the Banker, the first two remaining in their place. That which will appear evident, if we consider that in the six arrangements

$$\begin{matrix} bcadf & bcdaf & bcfad \\ bcafd & bcdfa & bcfda \end{matrix}$$

g or h being in front of one of the three letters a, d, f, h or g can be arranged in three different ways with the last two.

It is clear that it will be likewise of the three other columns of one hundred twenty arrangements where the first two letters could be bd, bg, bh. From all this it follows that we will have

$$S = \frac{3 \times 720A + 4 \times 3 \times \overline{3 \times 120 \times \frac{5}{3}A + 3 \times 36 \times 2A}}{5040} = A + \frac{8}{105}A.$$

3°. In order to find what is the lot of the Banker when the deck being composed of seven covered cards, the Punter takes one of them which is four times in these seven cards; we will observe, 1°, that imagining the five thousand forty possible arrangements of seven cards put on seven columns of seven hundred twenty arrangements each, of which the one begins with a, the second with b, &c. as above, there will be four of these seven which will give A to the Banker. 2°. By distributing each of the three others on six columns of one hundred twenty arrangements each, four of these six will furnish each one hundred twenty arrangements which will give $\frac{5}{3}A$ to the Banker, & the two others twenty-four arrangements each which will give to him $2A$. We will have therefore

$$S = \frac{4 \times 720A + 3 \times \overline{4 \times 120 \times \frac{5}{3}A + 2 \times 24 \times 2A}}{5040} = A + \frac{11}{105}A$$

It would be useless to pursue in detail the solution of a greater number of cases. We see rather by the preceding reflections, what will be those which would be necessary to make under the assumption that the Banker having nine covered cards, the Punter takes one of them. Thus, 1°, we will find that if the card of the Punter is two times in these nine cards, we will have

$$S = \frac{2 \times 40320A + 7 \times \overline{2 \times 5040 \times \frac{5}{3}A + 6 \times 2160 \times 2A}}{5040 \times 8 \times 9} = \frac{379680}{362880}A = A + \frac{5}{108}A.$$

2°. If the card of the Punter is three times in the nine cards of the Banker, we will have

$$S = \frac{3 \times 40320A + 6 \times \overline{3 \times 5040 \times \frac{5}{3}A + 5 \times 9360 \times 2A}}{362880} = \frac{284480}{362880}A = A + \frac{5}{84}A.$$

3°. Finally if the card of the Punter is four times in these nine cards, we will have

$$S = \frac{4 \times 40320A + 5 \times \overline{4 \times 5040 \times \frac{5}{3}A + 4 \times 1584 \times 2A}}{362880} = \frac{392640}{362880}A = A + \frac{31}{378}A.$$

It follows from that which precedes that if we name g the lot of the Banker with a number of cards expressed by $p-2$, p the number of cards, q the number of times that the card of the Punter is found in the deck, we have generally the lot of the Banker

$$= \frac{q}{p}A + \frac{p-q}{p} \times \frac{q}{p-1} \times \frac{5}{3}A + \frac{q.p-q.p-q-1.p-q-2}{p.p-1.p-2.p-3} \times 2A$$
$$+ \frac{p-q.p-q-1}{p.p-1} \times g - 1 \times \frac{q}{p-2}A - \frac{p-q-2}{p-2.p-3} \times q \times \frac{5}{3}A.$$

SECOND METHOD.

119. Let B be the lot of the Banker in the first deal, y his lot in the second, z his lot in the third, u his lot in the fourth, &c. A, the stake of the Punter, & the rest as above:

We find

$$\begin{aligned}
B &= \frac{q}{p}A + \frac{p-q}{p} \times \frac{q}{p-1} \times \frac{5}{3}A + \frac{p-q.p-q-1}{p.p-1}y. \\
y &= \frac{p-q-2\times q}{p-2.p-3} \times 2A + \frac{p-q-2.p-q-3}{p-2.p-3} \times z, \\
z &= \frac{p-q-4\times q}{p-4.p-5} \times 2A + \frac{p-q-4.p-q-5}{p-4.p-5} \times u, \\
u &= \frac{p-q-6\times q}{p-6.p-7} \times 2A + \frac{p-q-6.p-q-7}{p-6.p-7} \times t, \\
t &= \text{\&c.}
\end{aligned}$$

If we substitute for y, z, u & t their values, we will have this indefinite formula:

$$\begin{aligned}
B = \frac{q}{p}A &+ \frac{q.p-q}{p.p-1} \times \frac{5}{3}A + \frac{q.p-q.p-q-1.p-q-2}{p.p-1.p-2.p-3} \times 2A \\
&+ \frac{q.p-q.p-q-1.p-q-2.p-q-3.p-q-4}{p.p-1.p-2.p-3.p-4.p-5} \times 2A \\
&+ \frac{q.p-q.p-q-1.p-q-2.p-q-3.p-q-4.p-q-5.p-q-6}{p.p-1.p-2.p-3.p-4.p-5.p-6.p-7} \times 2A \\
&+ \text{\&c.}
\end{aligned}$$

which gives the lot of the Banker, whatever be the values of p & of q. And making on this formula the same reflections as we have already made on that of Pharaon, *art. 71*, we will find the demonstration of the rule which follows.

It is necessary to add to these two terms $\frac{q}{p}A + \frac{q.p-q}{p.p-1} \times \frac{5}{3}A$ *the sum of the figurate numbers, which in a horizontal row, of which the index is* q*, corresponds to some odd natural numbers,* Table I, art. I, *to begin with* $p-4$*; to multiply this sum by as many products of the natural numbers* $1, 2, 3, 4, 5, 6,$ *&c. as there are units in* q*; to multiply again by* $2A$*, & to divide by as many products of the quantities* p, $p-1$, $p-2$, $p-3$, *&c. as there are units in* q*: we will have the lot of the Banker.*

In order to prove these sums, & to deduce from this rule some particular formulas parallel to those which I have given for Pharaon, we will be served either from the property of the figurate numbers of which we speak on page 92, *art. 73*, or of the method, *art. 54*, & we will have the formulas which follow.

$$B \overset{2}{=} \frac{1}{3} \times \frac{p+1}{pp-p} \qquad C \overset{3}{=} \frac{pp-2p-3}{2p^3-6pp+4p} \qquad D \overset{4}{=} \frac{2pp-3p-11}{3p^3-9pp+6p} \qquad E \overset{5}{=} \text{\&c.}$$

The first B expresses the advantage of the Banker, when the card of the Punter is found twice in the deck; C his advantage when it is three times; D his advantage when it is four times; E his advantage if it will be found five times, &c.

If we wish a general formula such as I have given one for Pharaon, *art. 74*, we will have

$$\begin{aligned}
\frac{q\times p-q}{p.p-1} \times \frac{2}{3}a - \times\overline{p-q\times p-q-1} \times \frac{1}{2} \times \frac{q}{p.p-1.p-2}a \\
+ \frac{1}{4} \times \frac{q.q-1}{p.p-1.p-2.p-3} + \frac{1}{8} \times \frac{q.q-1.q-2}{p.p-1.p-2.p-3.p-4} \\
+ \frac{1}{16} \times \frac{q.q-1.q-2.q-3}{p.p-1.p-2.p-3.p-4.p-5} \\
+ \frac{1}{32} \times \frac{q.q-1.q-2.q-3.q-4}{p.p-1.p-2.p-3.p-4.p-5.p-6}
\end{aligned}$$

In order to have the advantage of the Banker, it is necessary to take as many terms of this series as there are units in q, with the exception which follows; that is namely that the last term, instead of being multiplied as the preceding negatives by $p-q$, $p-q-1$, must be multiplied only by $p-q-1$, when q is an even number; & by $p-q$, when q is an odd number; & that it must have in the denominator the same products as the term which precedes it: the origin & the demonstration of this formula is discovered without pain in that which we have given on Pharaon, *art. 73*.

The question in Bassette & in Pharaon is to find the sum of the figurate numbers of the arithmetic triangle, *art. 1*, interposed two by two, by beginning in Pharaon with that which corresponds to the even natural number $p-2$; & in Bassette, with the one which corresponds to the odd natural number $p-4$.

When q is not an odd number, that which happens only in the first hand, when the Banker turns the deck of cards: one is in the kind of Pharaon.

REMARK I.

120. The advantage of the Banker is expressed in the 1st column of the Table adjoined here by a fraction of which the numerator is always the number 2, & of which the denominator is always the product of these odd numbers $5,\ 7,\ 9,\ 11,\ 13,\ 15,$ by 3.

In the second column the numerators follow the order of the natural numbers $3,\ 4,\ 5, 6,$ &c. & the denominators are the same as in the first column, with the exception that it is necessary to imagine them multiplied by the series of numbers $2,\ 3,\ 4,\ 5,\ 6,$ &c. in such a way that if the numerators of the first column were $4,\ 6,\ 8,\ 10,\ 12,$ &c. the denominators of the two columns would be the same.

In the third column the numerators have for differences the odd numbers $5,\ 7,\ 9,\ 11,\ 13,$ &c. of which the constant difference is 2. The numbers of the denominator being divided by 3, are the pyramidal numbers taken two by two, of which the first difference is 75, the 2nd 72, & the 3rd which is constant, 24. These denominators can also be imagined to be formed in this manner $2\times4\times\overline{4-2},\ 3\times6\times\overline{6-3},\ 4\times8\times\overline{8-4},\ 5\times10\times\overline{10-5},\ 6\times12\times\overline{12-6}$, & thus in sequence.

In the fourth column the numerators have for first difference 42, & for constant difference 16. The denominators have for first difference 450, for second difference 432, & for constant difference 144.

REMARK II.

121. It will be easy to deduce from the order indicated above the same formulas that we have already found, without entering into any detail of the rules of Bassette. Thus in the case $q=3$, we will have the numerator

$$3+\frac{p-5}{2}\times 5+\frac{p-5.p-7}{2.2}\times\frac{2}{1.2}=\frac{pp-2p-3}{4};$$

& the denominator

$$=30+\frac{p-5}{2}\times 75+\frac{p-5.p-7}{2.2}\times\frac{72}{1.2}+\frac{p-5.p-7.p-9}{2.2.2}\times\frac{24}{1.2.3}=\frac{p^3-3pp-2p}{2}.$$

REMARK III.

122. In this Game, as in the one of Pharaon, the greatest advantage of the Banker is when the Punter takes one card which has not passed at all, & his least advantage is when the Punter take one of them which has passed twice; his advantage is also greater when the card of the Punter has passed three times, than when it has passed only one time.

REMARK IV.

123. In the game of Bassette the advantage of the Banker is less than in the game of Pharaon, that which we will understand easily by comparing the advantage of the Banker in the game of Bassette, when taking twelve cards the Punter takes one of them which is found either one, or two, or three, or four times, with his lot in this same case in the game of Pharaon.

We will find that the Punter putting a pistole on his card at Bassette, the advantage of the Banker will be 13 f. 4 d. when the card of the Punter will be four times in the twelve cards of the Banker, 12 f. 1 d. when it will be one time, 9 f. 8 d. when it will be three times, & 7 f. 3 d. when it will be twice; instead that in Pharaon the advantage is 19 f. 2 d. $\frac{10}{33}$ in the first case, 16 f. 8 d. in the second, 13 f. $7\frac{7}{11}$ d. in the third, & 10 f. $7\frac{3}{11}$ d. in the fourth, that which gives 3 liv. 1 denier advantage to the Banker for the four cases; instead that in Bassette the four together give only 2 liv. 2 f. 4 den. that which is just slightly less than two-thirds of the advantage of the Banker in the game of Pharaon.

REMARK V.

124. This game is presently much less in use than Pharaon. The cards which do not go, make losing in the game something of its vivacity. Besides there are often some disputes for knowing if the card of the Punter goes or does not go. We can not remedy these inconveniences, which are based on the nature of the game; but we would render this game more equal by agreeing that the faced cards paid only the half of the stake of the Punter, then the advantage of the Banker would be much less great, I have found that if the Banker would take only a third for the faces, this game would be disadvantageous to him. The greater part of the Remarks which we have made on the game of Pharaon, can take place in regard to this one, & it will not be useless to consult them.

Extract from the letter of Mr. Jean Bernoulli to Mr. de Montmort
From Basel this 17 March 1710

(p. 287) Besides that which I have said until here on the Game of Pharaon[1] must also extend on the one of Bassette, pag. 66 & the following,[2] or parallel to the calculation of the cases favorable & disadvantageous to the Banker, we can be spared the pain of examining minutely all the possible arrangements of the cards which can enter the hands of the Banker, by employing only the variations of the two letters a & b, as it had been done above, but we pass on to some others.

[1]See the extract from this same letter appended to the discussion of Pharaon.
[2]See § 116.

(p. 290) ...*page 64, l. 11 to end*[3], that which you said that the advantage of Pierre was $2A + \frac{16}{57}A$, is not correct, because I find only $A + \frac{16}{57}A$.

Extract from the Remarks of Mr. Nicolas Bernoulli
Appendix to the letter of Mr. Jean Bernoulli to Mr. Montmort
From Basel this 17 March 1710 (pg. 302–303)

Pag. 73 (first edition) Another formula. If $q < 1$, the gain of the one holding the cards is

$$= -\frac{1}{3}\frac{q.p-q}{p.p-1} + \frac{1}{2} \times \frac{q}{p} - \frac{1}{4} \times \frac{q.q-1}{p.p-q+1} + \frac{1}{8} \times \frac{q.q-1.q-2}{p.p-q+1.p-q+2}$$
$$- \frac{1}{16} \times \frac{q.q-1.q-2.q-3}{p.p-q+1.p-q+2.p-q+3} + \cdots$$

continuously to $\pm\frac{1}{2}q-1 \times \frac{q.q-1...2}{p.p-q+1.p-q+2...p-2}$ in A. If $q = 1$, you must add $\frac{1}{p} \times A$.

Pag. 74. In the Table, in the last case there is an error of calculation, for instance the gain of the one holding the cards when all 4 suits are hidden of all 52 inverted cards is not $\frac{2453842}{175592235}a$, but $\frac{454}{32487}a = \frac{2453870}{175592235}a$.

[3]See § 116.

opposite page 156

TABLE FOR BASSETTE

5	$1 := \frac{2}{15}$	$2 := \frac{3}{30}$	$3 := \frac{3}{30}$	$4 := \frac{24}{180}$
7	$1 := \frac{2}{21}$	$2 := \frac{4}{63}$	$3 := \frac{8}{105}$	$4 := \frac{66}{630}$
9	$1 := \frac{2}{27}$	$2 := \frac{5}{108}$	$3 := \frac{15}{252}$	$4 := \frac{124}{1512}$
11	$1 := \frac{2}{33}$	$2 := \frac{6}{165}$	$3 := \frac{24}{495}$	$4 := \frac{198}{2970}$
13	$1 := \frac{2}{39}$	$2 := \frac{7}{234}$	$3 := \frac{35}{858}$	$4 := \frac{288}{5148}$
15	$1 := \frac{2}{45}$	$2 := \frac{8}{315}$	$3 := \frac{48}{1365}$	$4 := \frac{394}{8190}$
17	$1 := \frac{2}{51}$	$2 := \frac{9}{408}$	$3 := \frac{63}{2040}$	$4 := \frac{516}{12240}$
19	$1 := \frac{2}{57}$	$2 := \frac{10}{513}$	$3 := \frac{80}{2907}$	$4 := \frac{654}{17442}$
21	$1 := \frac{2}{63}$	$2 := \frac{11}{630}$	$3 := \frac{99}{3990}$	$4 := \frac{808}{23940}$
23	$1 := \frac{2}{69}$	$2 := \frac{12}{759}$	$3 := \frac{120}{5313}$	$4 := \frac{978}{31878}$
25	$1 := \frac{2}{75}$	$2 := \frac{13}{900}$	$3 := \frac{143}{6900}$	$4 := \frac{1164}{41400}$
27	$1 := \frac{2}{81}$	$2 := \frac{14}{1053}$	$3 := \frac{168}{8775}$	$4 := \frac{1366}{52650}$
29	$1 := \frac{2}{87}$	$2 := \frac{15}{1218}$	$3 := \frac{195}{10962}$	$4 := \frac{1584}{65772}$
31	$1 := \frac{2}{93}$	$2 := \frac{16}{1395}$	$3 := \frac{224}{13485}$	$4 := \frac{1818}{80910}$
33	$1 := \frac{2}{99}$	$2 := \frac{17}{1584}$	$3 := \frac{255}{16368}$	$4 := \frac{2068}{98208}$
35	$1 := \frac{2}{105}$	$2 := \frac{18}{1785}$	$3 := \frac{288}{19635}$	$4 := \frac{2334}{117810}$
37	$1 := \frac{2}{111}$	$2 := \frac{19}{1998}$	$3 := \frac{323}{23310}$	$4 := \frac{2616}{139860}$
39	$1 := \frac{2}{117}$	$2 := \frac{20}{2223}$	$3 := \frac{360}{27417}$	$4 := \frac{2914}{164502}$
41	$1 := \frac{2}{123}$	$2 := \frac{21}{2460}$	$3 := \frac{399}{31980}$	$4 := \frac{3228}{191880}$
43	$1 := \frac{2}{129}$	$2 := \frac{22}{2709}$	$3 := \frac{440}{37023}$	$4 := \frac{3558}{222138}$
45	$1 := \frac{2}{135}$	$2 := \frac{23}{2970}$	$3 := \frac{483}{42570}$	$4 := \frac{3904}{255420}$
47	$1 := \frac{2}{141}$	$2 := \frac{24}{3243}$	$3 := \frac{528}{48645}$	$4 := \frac{4266}{391870}$
49	$1 := \frac{2}{147}$	$2 := \frac{25}{3528}$	$3 := \frac{575}{55272}$	$4 := \frac{4644}{331632}$
51	$1 := * \quad *$	$2 := * \quad *$	$3 := \frac{624}{62475}$	$4 := \frac{5038}{374850}$
52	$1 := * \quad *$	$2 := * \quad *$	$3 := * \quad *$	$4 := \frac{454}{32487}$