ESSAY

D'ANALYSE

SUR

LES JEUX DE HAZARD

Pierre de Montmort

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PREFACE

For a long time Geometers boast of being able by their methods to discover in the natural Sciences, all the truths which are carried to it by the human mind; & it is certain that by the marvelous mixture that they have made since fifty years of Geometry with Physics, they have forced men to recognize that that which they say on the advantage of Geometry is not without foundation. What glory would it be for this Science if it could yet serve to regulate the judgments & the conduct of men in the practice of the things of life!

The elder of the Messrs. Bernoulli so known both in the scholarly world, has not believed that it is impossible to carry Geometry up to this point. He had undertaken to give some Rules in order to judge the probability of future events, & of which the knowledge is hidden to us, either in Games, or in the other things of life where hazard alone takes part. The title of this Work must be *De arte conjectandi, the art of conjecturing.* A premature death has not permitted him to put the finishing touch.

Mr. de Fontenelle & Mr. Saurin have given each a short Analysis of this Book; the first in the History of the Academy; the other in the Journal des Sçavans of France. Here is, according to these Authors, what was the plan of this Work. Mr. Bernoulli divided it into four Parts; in the first three he gave the solution of diverse Problems on the Games of chance: one must find many new things with respect to infinite series, on combinations & the changes of order, with the solution of the five Problems proposed a long time ago to Geometers by Mr. Huygens. In the fourth Part, he employed the methods what he had given in the first three, to resolve diverse moral, political & civil questions.

Year 1705, page 148.
Year 1706, page 81.

One has not apprised us at all what are the Games of which this Author determined the divisions, nor what subjects of politics & morals he had undertaken to clarify; but as surprising as this project be, there is a place to believe that this scholarly Author had executed it perfectly. Mr. Bernoulli was too superior to the others in order to wish to deceive by it, he was of that small number of rare men who are appropriate to invent, & I am persuaded that it had held all that which the title of his Book promised.

Nothing delays more the advancement of the Sciences, & puts a greater obstacle to the discovery of hidden truths, than the mistrust which we have of our forces. The greater part of the things which appear impossible are only by lack of giving the human mind all the extent that it is able to have.

Many of my friends had prompted me, for already a long time, to test if Algebra could not at all reach to determining what is the advantage of the Banker in the Game of Pharaon. I had never dared to undertake this research, because I knew that the number of all the diverse possible arrangements of fifty-two cards, surpasses more than one hundred thousand millions of times that of the grains of sand that the globe of the earth could contain; & it appeared to me not possible to unmix, in a number so vast, the arrangements which are advantageous to the Banker, from among those which are contrary or indifferent to him. I would yet be under this prejudice if the success of the late Mr. Bernoulli had not invited me two years ago to seek the different chances of this Game. I was happier than I had dared to hope, because beyond the general solution of this Problem, I perceived the routes that it was necessary to take in order to uncover an infinity of similar, or even of much more difficult. I knew that one would be able to go further in this country where no person had yet been; I will flatter myself that one could make an ample harvest of truths equally curious & novel: this will gave me the thought of working at foundation on this matter, & the desire to compensate in some way the Public of the loss that it had if it was deprived of the excellent Work of Mr. Bernoulli. Diverse reflections have confirmed me in this design.

It is particularly in the Games of chance that the feebleness of the human mind appears & the tendency that it has to superstition. Nothing is so ordinary as seeing some Players attribute their misfortune to the persons who approach them, & to other circumstances which are no less indifferent to the events of the game. There are of them who make it a law of taking only the cards which win, in the thought that a certain fortune is attached to them. Others on the contrary are attached to taking the losing cards, under the opinion that having lost many times, it is less probable that they will lose again, as if the past could decide some thing for the future. There are of them who affect certain places & certain days. One sees of them who refuse to shuffle the cards, except in certain situations, & who could believe to lose infallibly if they were themselves deviated in that from their rules. Finally the greater part seek their advantages where they are not, or else they neglect them entirely.

One is able to say very nearly the same thing of the conduct of men in all the actions of life where chance has some part. These are the same prejudices which govern them, it is imagination which regulates their processes, & which gives birth blindly to their fears & their hopes. Often they abandon a small quite certain in order to pursue daringly after a greater benefit, of which the acquisition is as impossible; & often by too much mistrust they renounce some considerable & well founded expectations, in order to conserve a benefit of which the value has no proportion at all with the one they neglect. The general principle of these prejudices & of these errors is that the greater part of men attributing the distribution of the good & the bad, & generally all the events of this world to a fatal power which acts without order & without rule, they believe that it is worth so much to be abandoned to the blind Divinity, who one names Fortune, than to force her to be favorable to them by following the rules of prudence which appears imaginary to them.

I have therefore believed that it would be useful, not only to the Players, but to men in general, to know that chance has some rules which are able to be known, & that failing to know these rules they make mistakes every day, of which the unfortunate series must be attributed to them with more reason than to the destiny that they accuse. I could report in proof an infinity of examples drawn either from Games, or from other things of life of which the event depends upon chance. It is certain that men are not served enough by their mind in order to obtain that which they desire even with the most ardor, & that they do not make enough effort in order to remove from Fortune that which they could subtract by the rules of prudence.

One has believed that this matter would prompt the curiosity of even those who have the least of it for abstract knowledge. One loves naturally to see clear in that which one does, even independently of all interest. One would play without doubt with more agreement if one could know at each coup the expectation that one has to win, or the risk that one incurs to lose. One would be more tranquil on the events of the game, & one would sense better the ridicule of those continual complaints to which the greater part of the Players let themselves go in the most common encounters, when they are contrary to them.

If the exact knowledge of the chances of the game does not suffice alone to the Players in order to make them win, it is able at least to serve to make them take the better part in the doubtful things, &, that which is very important, to teach them up to what point the conditions of certain Games that greed & idleness introduce every day are disadvantages for them. For me I believe that if the Players knew that when they put into Pharaon a louis of three livres on a card which has passed three times, the stock being no more than twelve cards, they make precisely the same thing as if they gave as pure gift one livre one sol & eight deniers to the Banker, there would be few of them who wanted to tempt Fortune with so much disadvantage.

The conduct of men makes most often their good or their bad fortune, & wise people give to chance the least that they can.

We cannot know the future, but we can always in the Games of chance, & often in the other things of life, know with exactitude how much more probable it is that a certain thing will arrive in such fashion rather than another thing! And since these are there the limits of our knowledge, we must at least try to reach there.

Everyone knows that by lack of evidence, we must seek the probability in order for us to approach the truth; but one knows not at all rather that there are some probabilities greater & lesser to infinity, & that the mind in order to be a good judge, must distinguish all the degrees of them, since it often happens that a thing being uncertain, it is nonetheless certain & even evident that it is probable, & more probable than every other.

It appears that one has not learned enough to the present that one can give some infallible rules in order to calculate the differences which are found between diverse probabilities.

We have wished to give in this Work an essay of this new art, by applying it in a manner that which has been until now in a great obscurity, & which appears susceptible to no precision. We have believed that it was more proper than every other to give from the esteem for Analysis, this marvelous art which is the key of all the exact Sciences, & which is apparently neglected only because one did not know enough the extent of its uses; because instead that one has employed until now Algebra & Analysis only to uncover the constant & immutable ratios between some numbers & some figures, one is served here in order to discover the ratios of probability between some uncertain things & which have nothing fixed, that which seems quite opposed to the spirit of Geometry, & in some fashion beyond its rules. It is this that the illustrious Author of the History of the Academy senses judiciously at the place that I have already cited. *It is not so glorious*, says he, *to the spirit of Geometry to regulate in Physics than in the moral things, so casual, so complicated, so variable. The more a matter is opposed to it & rebels, the more it has honor to reckon it.*

I divide this Treatise into three Parts: in the first I give the solution of diverse Problems on the Games of cards which are in use; I examine first those which are of pure chance, such as Pharaon, Bassete, Lansquenet & Treize; I determine what is the advantage or the disadvantage of the Players in all the possible circumstances of these Games. Geometers will find in the solution of these Problems all the generality that they are able to wish; & the Players will encounter some quite singular novelties, of which it is important to be instructed. One is limited to examine these four Games in order to not make too large a Volume, & one has preferred them to the others since they are more in use, & since they have appeared to me most curious. I give next diverse Theorems on combinations, of which the greater part are new & of great use; I apply them to resolve many particular Problems on Hombre, Piquet, Imperiale, Brelan, &c. I have not been able to treat these last Games with the same extent as the preceding, I render reason on pages 51 to 53.

The second part contains a general solution of all the questions that one can propose on Quinquenove, the Game of three dice, & the Game of chance. The first two are the only Games of dice which are in use in France, the last in known only in England. I give next some rules in order to play the most perfectly as it is possible a Game of which the invention is ingenious, & which holds equally on two Games of cards Her & Tontine. The person who has apprised me of this Game has not been able to say how one names it. In order to not leave it without name, I have called it the Game of Expectation. One will find also the solution of some rather easy Problems on the Game of Trictrac. There is one which can be of some use for the Players.

I end this second Part with a very general Problem on dice. This Problem with the one of Proposition 14, contains all the theory of combinations. I add three Problems so to serve as Example, there is one of them on the Game of the first Rafle; the second is on the three Rafles counted; the last is on a Game of which the Baron of la Hontan makes mention in the second Book of his Voyages, & which he says to be quite in use among the Savages of Canada. The name of it is not magnificent, it is called the Game of the Nuts.

One will be able to note that all the different Games of dice that one examines in this second Part, give the disadvantage to the one who holds the die, instead that in the Games of cards such as Pharaon, Bassete, Lansquenet, Treize, the one who holds the cards has a considerable advantage.

It is to believe that those who have invented these Games have not claimed at all to render them entirely equal; or, that which appears more probable, that they have not known enough the nature in order to well distribute the chances. In the greater part the conditions are so unequal for the Players, that one would be well founded to sustain that one can not win with justice, as without doubt one can lose without being duped.

Although in this Treatise I have much more in view the pleasure of the Geometers than the utility of the Players, & that according to us those who lose their time in the Game merit well to lose their money, I have not neglected in uncovering the advantage or the disadvantage of the Players, to remark in what manner it would be necessary to reform the Games in order to render them perfectly equal.

In the third Part I give the solution of the five Problems proposed by Mr. Huygens, & I add some others to them, which although less curious & less difficult than those which are contained in the two Parts preceding, do not abandon having their utility with respect to this matter. I terminate it by proposing, in the imitation of Mr. Huygens, four rather singular Problems. But I believe the Geometers must avert who would have the curiosity to attempt the solution of them, that they will find no less difficulty than in the most difficult problems of the integral calculus. Those who would regard these questions only as some Problems of Arithmetic, will recognize that if they suppose less knowledge in Geometry, they demand perhaps more skill, & certainly much more exactitude & circumspection.

If I myself had proposed to follow in all the project of Mr. Bernoulli, I ought to add a fourth Part, where I had made application of the methods contained in the first three, to some political, economic or moral subjects. That which has prevented me, it is the difficulty where I myself am found to make some hypotheses which being applied onto some certain facts, could lead me & sustain me in my researches: But not having the convenience to satisfy myself entirely above, I have believed that it is worth more to remit this work at another time, or to leave the glory to some person other more able than me, than to say some things either too common or not very exact, which have not responded to the attention of the Reader & to the beauty of the subject. I will limit myself to remark at least in words that it will be possible for me, the relation that there is between this material & that of the Games, & the view that it would be necessary to take in order to reuse.

Exactly speaking nothing depends upon chance; when one studies nature, one is soon convinced that its Author acts in a general & uniform manner which bears the character of a wise & of an infinite prescience. Thus in order to attach to this word *chance* an idea which is conformed to the true Philosophy, one must think that everything being regulated according to certain laws, of which most often the order is not known to us, the former depend on chance of which the natural cause is hidden from us. According to this definition one is able to say that the life of man is a Game in which chance rules.

In order to see more precisely that the Analysis of the Geometers, & principally that which one uses in this Treatise, is proper to dissipate in part the darkness which seems to spread over the things of civil life which have relation to the future, it is necessary to remark that likewise that there are some Games which are regulated by chance alone, & others which are regulated in part by chance & in part by the skill of the Players; thus among the things of life there are of them of which success depends entirely on chance, & others in which the conduct of men has much part; & that generally in all things of life on which we have to take our part, our deliberation must be reduced, as in the wagers on the Games, to comparing the number of cases where a certain event will happen, to the number of cases where it will not happen; or, in order to speak in Geometry, to examine if that which we hope multiplied by the degree of probability that there is that we will obtain it, equals or surpasses our stake, that is the advances which we must make, either pain, or money, or credit, &c.

It follows thence that the same rules of Analysis which have served us to determine in the Games the divisions of the Players & the manner of which they must conduct their Game, can also serve to determine the just degree of our hopes in our diverse enterprises, & teach us the the conduct that we must hold in order to find the most advantage that it is possible. It is clear, for example, that the same method which has served us to determine under what circumstances it is proper in Hombre to renounce the cards without taking in the hope to make the slam, is able to be employed, although more difficultly, in order to determine under what circumstances of life it is necessary to sacrifice a little wealth in the hope of obtaining a greater.

In order to continue this comparison, it is necessary to remark that the same reasons which prevent us from being able to resolve all the questions that one is able to propose on Games, prevent also that one is able to resolve those that one is able to propose on the things of civil life. These reasons are of two sorts: the first is the uncertainty where we are of the division that those will take in our regard of which the actions must regulate the event of our enterprises. The impact of a body decides & of the route that it must hold & of the speed that it must have, because the laws of the communications of the movements are fixed and invariable. But the reasons & the different motives that men are able to have in order to act in one fashion rather than another, are not able to assure us on what side they will be determined. Often they know not at all their interests, & often they do not follow them when they know them, caprice determines them much more than reason, & it is always to guess than to wish to judge on that which depends on the liberty of men.

The second cause of our ignorance in the things which depend on the future, is based on this that the limits of our mind being quite distant, all the knowledge which supposes too great a number of relations are beyond its forces. Now in many Games, & in the greater part of the things of life, the comparisons that it is necessary to make are in such great number, that it is nearly impossible to exhaust them.

To determine how much the die is worth between two equal Players of Trictrac, & how much the hand is worth in Piquet, what piece is most advantageous in the Game of Chess of the bishop or of the knight, & how much one is better than the other; these are thence some Problems of which I hold the solution impossible to men. It is likewise for it, & for the same reasons of the greater part of the questions of morals and politics: for example, to determine if in such & such circumstance I must have more regard to the recommendation of a parent, than to the prayer of a certain number of friends: If a certain commerce is advantageous or prejudicial to a Nation; what must be the success of a negotiation & of a military exercise, &c.

Insurance contracts which are so common among Merchants, principally in the Republics, do not always enrich the Assurers; & the most able Policies of England experience every day to their loss in these gross wagers that one makes in that country on the events of the war, that the prudence of men is insufficient to penetrate surely into the future.

It is true that with much justice of mind, a great knowledge of facts, & especially from the secret motives which the oscillation & movement to the affairs give, one is able to discover with enough probability what is the better division in these wagers: but it is impossible to ever come to the point of being able to determine by the exact ratio of two numbers, how much one part is better than the other.

Whatever help that the human mind is able to receive from Geometry, this virtue, that one names prudence, will never have but some uncertain rules; for a small number of truths & certain principles that politics & morals contain, one finds an infinity of obscurities impenetrable to the human mind.

Each usage that one is able to draw from Geometry with respect to these sorts of Problems, consists in this that one is able to assure that those who will be rendered familiar the kind of Logic of which one makes use in this Treatise, will be more proper to discover in it the different degrees of probability in the diverse parts that one is able to take on the things which regard morals, or on those which have relation to civil life, & to avoid the error in their judgments by the habits that they will have acquired to distinguish the true from the probable, & to give their consent only to the evidence.

Since men think that which they would wish, it is certain that this force & this justice of mind that one acquires in the research on the abstract truths, extends also to the sensible truths, & so to say practice. Analysis is an instrument which serves to all when one knows to handle it well. All the truths are held among them, & when one has made some time a test of its forces on the exact notions that we have of the numbers & of the extent, one uses them with more success on the less exact knowledge which is able to be object of our mind. Those who have written best on Metaphysics, Physics, perhaps even on Medicine & on Morals, were excellent Geometers. Experience therefore ought to convince of the utility of Geometry those to whom the ratios are not able to persuade.

In order to terminate this parallel between the Problems on the Games, & the questions that one is able to propose on economic, political and moral things, it is necessary to observe that in these latter as in those of the Games, there is a kind of Problems that one will be able to resolve by observing these two rules; (1) to limit the question that one proposes to a small number of assumptions, established on certain facts; (2) to set aside all the circumstances to which the liberty of man, this perpetual peril of our knowledge, would be able to have some part. It is to believe that Mr. Bernoulli had regard to these rules in the fourth part of his Work, & it is certain that with these two restrictions one would be able to treat many subjects either of politics or of morals with all the exactitude of geometric truths.

It is this that Mr. Halley has made admirably in a Memoir which is found in the Philosophical Transactions of England, num. 196, where this scholarly Englishman undertakes to determine the degree of mortality of human kind. This piece is full of curious things, of which the Reader would see the extract here with pleasure: But this Preface being perhaps already too long, I will report from it only one which is treated by the Author with much finesse. It is a method to determine on what footing annuities must be regulated. He gives a Table entirely calculated for the different ages from five to five years from one to seventy. This Table shows how much was the advantage to the English the wager that King William made them then, by giving 14 percent per year of life-annuity, that which is very nearly the seventh part of the fund. One sees by this Table that a person aged ten years should have only the thirteenth part, & a man aged thirty-six years the eleventh, & finally that the interest of ten percent is due only to the persons aged forty-three to forty-four years. He pushes this idea yet further, & he examines on what footing a life-annuity must be regulated which would be on the head of two or many persons of different ages.

This matter appears exhausted in the Memoir of Mr. Halley. One finds some other similar ways happily enough, although with less exactitude, in the political Arithmetic of Sir Perry. But there remains many others of this nature that one would be able to treat with the same success & the same utility for the Public.

I believe I must speak now of two illustrious Geometers to whom I owe the first views that I have had on the subject that I treat. In 1654 Mr. Pascal resolved this Problem: *Two persons play in a fair game to a certain number of points; one of the two is supposed to have more points than the other: One demands how they must divide the money in the Game in case that they wish to interrupt the game without finishing it*. One is able to see the solution of this Problem in a very short Book that one has found printed after his death, & which has for title *Arithmetic Triangle*. This great Man who had much meditated on the properties of numbers, made diverse applications of this triangle to the rules of divisions & to the combinations.

The Chevalier de Meré had proposed to him this Problem, he had also proposed to him some others on dice: For example, to determine in how many trials one is able to bring forth a certain rafle, & some other of this sort easy enough. This Chevalier, who had much good mind as Geometer, resolved the Problems on the dice, but neither he nor Mr. de Roberval were able to resolve the one of the parts. Mr. Pascal proposed it to Mr. Fermat with whom he was in commerce in friendship & Geometry, & who in that Science was only inferior to Mr. Descartes.

This Geometer arrived to the solution of the Problem by a different way from that of Mr. Pascal: He will go even further, & he will assure that his method was general for such number of Players as there were. Mr. Pascal did not believe that it was able to be led there, & he tried to show him in a letter that is found with some others on this subject in the posthumous Works of Mr. Fermat printed at Toulouse, that his method, which he recognized good for two Players, is not correct for a greater number. On sees not at all in this Compilation the response of Mr. Fermat, but it is certain that right was on his side; his method is sure, & is extended to such number of Players as there be.

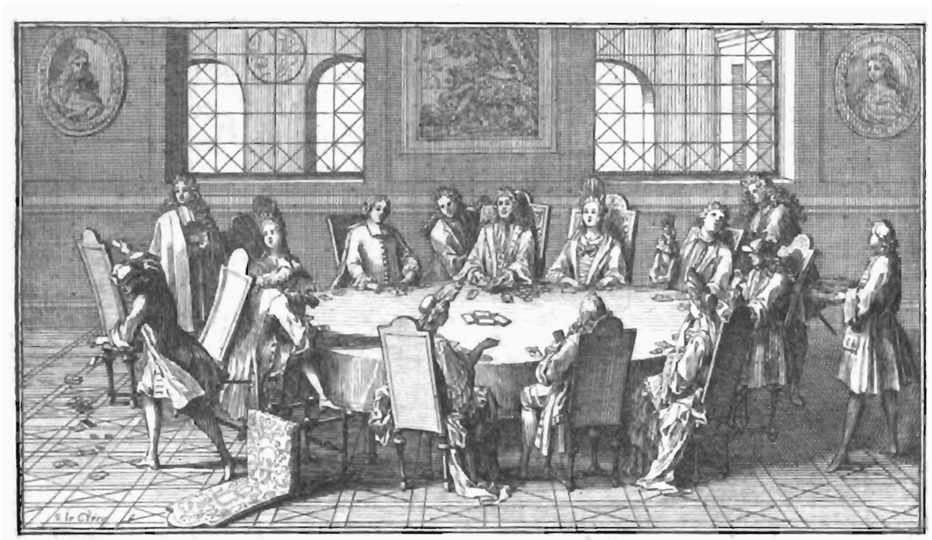
Very near this time Mr. Huygens, this famous Geometer who has enriched all the parts of Mathematics with so many good discoveries, having heard speak of these Problems, undertook to resolve them, & employed, in order to come to the end of them, the analytic method, which for the ordinary leads further than every other. He made from these Problems a small Latin Treatise which comprises around one sheet, & is found at the end of the Book of Mr. Shotten, entitled, *Exercitationes Geometricae*.

Although this Author undertook to determine the parts of the Players in no game of cards nor of dice, & although he is limited to that which he has most easy in this matter, & nearly to the sole Problems of Mr. Pascal, one sees by the letter he wrote to Mr. Schotten, who he valued much that which he gave in this small Work. *Nothing is more glorious*, says he, *in the art by which we make use in this Treatise, than to be able to give rules to some things which were dependent on chance, seems to recognize none of it, & thence to be subtracted from human reason*. And he adds: *I am assured that those who know to judge things, will recognize in reading this writing, that the subject is more serious & more important than it appears, that one puts the fundamentals of a very good & very subtle theory, & that the researches of Diophantus, which has for object only some abstract properties of numbers, are both easiest, & less agreeable than those that one is able to propose in this matter.*

The Author at the end of this Treatise invites the Geometers to research five Problems, of which none that I know have yet been resolved. There are three of these five of which he gives the solution, but with neither analysis nor demonstration, & he gives not at all the solution of the others.

As it is principally for the Geometers that I have composed this Treatise, & since ordinarily Scholars are not Players, I have believed I must explicate quite at length the Games of which I speak in this Work, & I have tried to omit no necessary circumstance. I myself was proposed first to put into ordinary language the solution of some of the easier Problems, such as are those of the third Part: but I have been constrained to abandon this design, in order to not be obliged to make some unending discourse that no person would have had the patience to follow. The usage of Algebra is to represent to the mind a great number of ideas under some quite short expressions, & to furnish great facilities in order to traverse with promptness the relationships of the things that one considers. I have believed that not wishing at all to make a large Book, I must not renounce this advantage; I am myself attached to explicate myself in such manner in the conclusion to each Problem, & in the Corollaries & the Remarks which are at the end of each solution, that I was able to make understood by everyone, & even the Players.

As one writes only in order to be read, I have tried to render easy the reading of this Work, & I have preferred without pain the satisfaction of the Reader to the esteem of these mediocre minds, who admire only that which costs much pain to them, & that which to them appears above their range. One will find that I myself am quite extended in the places that I have believed difficult, & principally in those which must spread their light on many truths. But as I know also that the utility of a Book of Mathematics consists less in the truths that it uncovers than in the disposition that it gives to the mind in discovering parallels of it, & that one acquires much more this disposition by finding that which the Author has already found, than in following it step by step, I have believed that I must not trouble myself in explicating all in detail, & even to demonstrate all, & that it seems to me to leave no difficulty except of which one is able to find the solution with a sufficient application. Finally I myself am proposed to save to the Reader the work of invention, & to leave in some sort the pleasure of it.



PROBLEMS
ON
THE GAMES OF CHANCE

FIRST PART

DEFINITION I.

In Games, Wagers & Lotteries, money that a Player risks is no longer counted to belong to him, because he has quitted the property; but in return he acquires a certain right on the fund of the Game, that is, to the money of the wager.

When the conditions of the game are equally advantageous to the Players, as in Pass-ten, & a small number of other Games, this right or expectation that it furnishes is equivalent to the stake of each of the Players. But in the Games, of which the conditions are unequally advantageous to the Players, such as are the greatest number, this right no longer corresponds exactly to the stake of the Players; & in this case, if they wish to be removed & to quit the game, in order to return to them the property of this thing, by renouncing that which chance would have given them, they must no longer divide equally the money of the game, but they must take a part more or less great, according as there is more or less probability that one or the others will win the entire sum of which one is agreed.

This supposed, if we name a the money of the Game, I will say that the *lot* of each Player is the just degree of expectation that he has to obtain a; & I will call, *division*, the convention or the settlement that the Players must make among them, when they wish to be removed without incurring the risk of the event of the Game; so that it is entirely equal to them, either to continue the game, or to disrupt it.

Thus, by supposing that two Players are agreed to chance each one half pistole at heads or tails, if we name the pistole a, I will say that the lot of each of the Players is $\frac{1}{2}a$; & that if changing with warning they wish to quit the Game, the division that they must make with one another, it is to remove each their half pistole.

DEFINITION II.

If two Players wish to play with neither advantage nor disadvantage in a Game of which the conditions are unequal, it is necessary that the one, to whom they are favorable, set into the Game more than the other; & in order to speak with precision, it is necessary that his stake be to that of the other Player in the same ratio as the diverse degrees of expectation that they have to win. If they play end to end, it is clear that the advantage is for one of these Players, & that it is necessary to understand by this word, *advantage*, the excess of that which he awaits from chance over that which he put into the Game. For example, if one supposes that Paul wagering end to end an écu against Pierre, to bring forth a doublet on the first coup with two dice, one has found for the lot of Pierre $A + \frac{2}{3}A$, A designating an écu, this fraction $\frac{2}{3}A$ which is the excess of the expectation or of the lot of Pierre over his stake which is A, will express his advantage, or that which Paul should give to Pierre, if after having made this convention with him, he wished to disrupt the wager, since by virtue of the condition of this wager, Pierre has no less right on the two-thirds of the écu of Paul, than he has on the écu which he has set into the game.

For it is necessary to remark that although it is very uncertain if Paul will win or will not win, & that there is no contradiction at all that he win a thousand times in sequence, it is nevertheless very certain that in order to buy the right of Pierre it would be necessary to give to him forty sols; & that if Paul was obliged to play three trials under the preceding conditions, Pierre would be able to count as well on two écus of profit as on two écus that Paul would have given to him in pure gift, on condition that he wished to play three écus against him at heads or tails.

Although these terms *advantage* & *disadvantage* seem to be clear, because they are common & familiar, I have believed that it was proper in order to remove all equivocation, to explicate in what manner I understand them; it has seemed to me that nearly everyone attached false ideas there.

PROPOSITION I.

LEMMA.

The number of chances which are able to make Pierre win, & give to him A, being m; & the number of chances which are able to make him lose or give to him zero, being n, I say that if there are only these two kinds of chances, & if one understands by A the money of the game, one will have the lot of Pierre $= \frac{mA+a\times 0}{m+n}$.

In order to prove it, let x be the lot of Pierre, y the one of the other Player, who I will name Paul, one will have $x + y = A$. One will have also $x.y :: m.n$, for the lot of each of these Players is as their expectation, & this expectation is proportioned to the facilities or to the means that they have to win, that is to the number of trials which will give A to them. From these two equations $y = \frac{nx}{m}$ & $x + y = A$, one will draw $x = \frac{Am}{m+n}C$. QED.

Thus supposing, for example, that Pierre wagers against Paul to bring forth a 6 on the first coup with a die, his lot will be $\frac{1\times A+5\times 0}{1+5} = \frac{1}{6}A$, & the lot of Paul will be $\frac{5}{6}A$: Whence it follows that in order to wager equally, Pierre should put an écu into the game, against Paul five écus, since in an equal wager the stakes of two Players must have the same ratio as the diverse degrees of probability or of expectation that each of the Players has to win.

I would have been able to announce this Lemma more generally, the demonstration would have been the same, but I have feared to render obscure a thing which appeared to me of the greatest evidence, namely that the lot of Pierre is the ratio of all the coups which are favorable to him to the number of all the possible coups; or, if one wishes, that his lot is the ratio of the degree of expectation or of facility that he has to win, to the risk that he incurs to lose.

PROBLEM
ON PHARAON

To determine generally the advantage of the Banker with respect to the Punters.

The principal rules of this game are, (1) that the Banker deals with an entire deck composed of fifty-two cards, (2) That the Banker draws all the cards in order, putting the ones to the right, & and the others to his left, by commencing with the right. (3) That in each hand, or in each deal, that is to say of two by two cards, the Punter has the liberty to take one or many cards, & to venture over a certain sum. (4) That the Banker wins the stake of the Punter, when the card of the Punter arrives in the right hand in an odd rank, & that he loses, when the card of the Punter falls to the left hand and in an even rank. (5) That the Banker takes the half of that which the Punter has set on his card, when in the same deal the card of the Punter comes twice, that which makes a portion of the advantage of the Banker. And finally, that the last card which should be for the Punter, is neither for him nor for the Banker, that which is again an advantage for the Banker.

It is evident that the conditions of this game are advantageous to the Banker. The difficulty is to determine this advantage, because it changes, & according to the number of cards that the Banker holds, and also according as the card of the Punter either has not passed, or has passed one or many times.

(1) The card of the Punter being only one time in the stock, the difference of the lot of the Banker and the Punter is founded on this, that among all the diverse possible arrangements of the cards of the Banker, there are of them a greater number which make him win, than there are of them which make him lose, the last card being considered as null; & in this case it is easy to notice that the advantage of the Banker increases, in measure as the number of cards of the Banker diminish.

(2) The card of the Punter being twice in the stock, the advantage of the Banker is drawn from the probability that there is, that the card of the Punter will come twice in the same deal; because then the Banker wins the half of the stake of the Punter, the sole case excepted where the card of the Punter would be in doublet in the last deal, that which would give to the Banker the entire stake of the Punter.

(3) The card of the Punter being either three or four times in the hand of the Banker, the advantage of the Banker is founded on the possibility that there is, that the card of the Punter is found twice in the same deal, before it has come in pure gain or pure loss for the Banker. Now this possibility increases or diminishes, & according as there iis more or fewer cards in the hand of the Banker, & according as the card of the Punter is found there more or less times. From all this it follows that in order to know the advantage of the Banker with respect to the Punters in all the different circumstances of this Game, it is necessary to discover in all the different possible arrangements of the cards that the Banker holds, & and under the supposition that the card of the Punter is found, either one, or two, or three or four times, what are those which make him entirely win, what are those which give to him the half of the stake of the Punter, what are those which make him lose, & finally what are the arrangements which are neither winning nor losing.

In order to resolve this problem, it is proper to begin with the simplest case, & next passing to some more compound cases, it is necessary to seek some uniform law, & some analogy which is able to serve to disentangle from all the possible cases, the arrangements which are advantageous to the Banker, those which are indifferent to him, & finally those which are unfavorable to him. This method is the only one that one is able usefully to put into use, when one has as in this Problem, a very great number of comparisons to make. It is also the most natural way, in order to lead the mind of the Reader in these matters, & in order to prepare a general solution to it.

PROPOSITION II.

First Case.

On supposes that there remain four cards in the hands of the Banker, & that the one of the Punter is in it a certain number of times. The concern is to determine what is the lot of the Banker & that of the Punter: For example, if there is one écu for the card of the Punter, one demands what portion of the écu the Punter must give to the Banker in order to buy the right of his withdraw, & to not incur the risk of the game; or, that which reverts to the same, what is in this case the disadvantage of the Punter, in playing end to end against the Banker.

It is necessary to divide all that which all the diverse possible arrangements of the four cards give of gain or of loss to the Banker by the number of all these arrangements; the exponent of this division will express his lot.

In order to discover these different arrangements, it is necessary to observe that two letters, a & b, are able to be arranged in two ways, ab, ba; that three letters, a, b, c, are able to be arranged in six different ways: that which is seen by setting c into ab, & ba in all the places that it is able to be, namely, in the first, in the second, & in the third. These six arrangements are:

$$\begin{matrix} abc & bac & cab \\ acb & bca & cba \end{matrix}$$

One will find likewise that four letters, a, b, c, d, are able to be arranged in twenty-four different ways, since d is able to occupy four different places in each of the six preceding arrangements.

Generally, if one names p the number of letters that one wishes to arrange in all the possible ways, q the number of all the diverse possible arrangements of a number of letters expressed by $p-1$, pq will express in how many different ways one is able to arrange some letters of which the number is expressed by p. For example, if one wishes to know in how many different ways four letters are able to be arranged, one will have by substituting for p its value four, & for q its value six, $pq = 24$, & this number will express in how many different ways four letters are able to be arranged.

This supposed, if one wishes to express the four cards of the Banker by the letters a, b, c, d, one will have all the different arrangements of four cards represented in the following Table.

$$\begin{matrix} abcd & bacd & cabd & dabc \\ abdc & badc & cadb & dacb \\ acbd & bcad & cbad & dbac \\ acdb & bcda & cbda & dbca \\ adbc & bdac & cdab & dcab \\ adcb & bdca & cdba & dcba \end{matrix}$$

(1) If one supposes that the card of the Punter designated by the letter a, is one time in the four cards of the Banker, & that the Punter has set on his card a sum of money expressed by A, one will note in considering the preceding Table, that there are twelve arrangements which give $2A$ to the Banker, six which make him lose or which give to him 0, & six which are indifferent to him.

Those which are the winning are:

$$\begin{array}{ll} abcd & bcad \\ abdc & bdac \\ acbd & cbad \\ acdb & cdab \\ adbc & dbac \\ adcb & dcab \end{array}$$

Those which make him lose are:

$$\begin{array}{lll} bacd & cabd & dabc \\ badc & cadb & dacb \end{array}$$

Thus expressing the sought lot by the letter s, one will have

$$s = \frac{12 \times 2A + 6 \times 0 + 6 \times A}{24} = \frac{5}{4}A = A + \frac{1}{4}A.$$

(2) If one supposes that the card of the Punter is found twice among the four cards of the Banker, & that the two letters a & b express that of the Punter, one will find that of the twenty-four arrangements of the Table, there are twelve which give $2A$ to the Banker:

$$\begin{array}{lll} acbd & bcad & cdba \\ acdb & bcda & cdab \\ adbc & bdac & dcba \\ adcb & bdca & dcab \end{array}$$

Four which give to him $\frac{3}{2}A$, that is to say, his écu and the half of the one of the Punter:

$$\begin{array}{ll} abcd & bacd \\ abdc & badc \end{array}$$

Eight which make him lose:

$$\begin{array}{ll} cabd & dabc \\ cadb & dacb \\ cbad & dbac \\ cbda & dbca \end{array}$$

Thus one will have $s = \frac{12 \times 2A + 4 \times \frac{3}{2}A + 8 \times 0}{24} = \frac{5}{4}A = A + \frac{1}{4}A.$

(3) If one supposes that the card of the Punter is found three times among the four cards of the Banker, & that the three letters a, b, c, express that of the Punter, one will find again the lot of the Banker$= A + \frac{1}{4}A$; because there are twelve arrangements which give

to him $\frac{3}{2}A$.

$$\begin{array}{ccc} abcd & bacd & cabd \\ abdc & badc & cadb \\ acbd & bcad & cbad \\ acdb & bcda & cbda \end{array}$$

Six which give to him $2A$:

$$\begin{array}{ccc} adbc & bdac & cdab \\ adcb & bdca & cdba \end{array}$$

Six which make him lose:

$$\begin{array}{ccc} dabc & dbac & dcba \\ dacb & dbca & dcab \end{array}$$

One will have therefore $s = \frac{12\times\frac{3}{2}A+6\times 2A+6\times 0}{24} = A + \frac{1}{4}A$.

(4) Finally it is evident that if the card of the Punter is found four times in the four cards of the Banker, the lot of the Banker will be $= A + \frac{1}{2}A$.

Corollary I.

It seems by the solution in this first case, that if the stake of the Punter is one écu, he must give fifteen sols which is the fourth of it to the Banker, in order to buy the right of his withdraw, this is if his card is one time, or two times, or three times in the four cards of the Banker.

Corollary II.

There would be an infinite labor to seek the other cases in the manner that one has resolved this one in searching in some Tables the favorable & contrary arrangements; because the number becomes immense in a greater number of cards; thus have I put the preceding solution, only in order to make me more easily understood in the following.

In order to resolve the preceding case in a methodical manner, & in order to discover the chances by the mind's view, it is necessary to remark,

That if the card of the Punter was one time in two cards, the lot of the Banker would be $\frac{3}{2}A$: Because of the two possible arrangements of two letters, there is one of them which gives $2A$, & one which gives to him A; & if the card of the Punter being in it more than one time, the lot of the Banker would be $2A$, that which is evident.

It is necessary to observe next that the card of the Punter being one time in four cards, if one places the twenty-four possible arrangements of four letters on four columns, of which the first begins all with a, the second with b, the third with c, the fourth with d, the first column will give $2A$ to the Banker in all its arrangements.

And when partitioning each of the three other columns into three columns of two arrangements, the one of these last three, namely the one where a occupies the second place, will give twice zero to the Banker, & each of the two last others will give to the Banker the same lot as he would have in the case that the Banker holding two cards, the one of the Punter is found one time there, that is to say $\frac{3}{2}A$, that which gives the lot of the Banker, as before

$$= \frac{1\times 6\times 2A + 3\times 2\times 2\times \frac{3}{2}A}{24} = \frac{30}{24}A = A + \frac{1}{4}A.$$

One will observe similarly that the card of the Punter expressed by the letters a & b being twice in the four cards, if one imagines the twenty-four different arrangements that the four cards are able to receive, put on four columns, as before, the two columns which commence with the letters a & b, will contain each four arrangements which will give $2A$ to the Banker, & two arrangements which will give $\frac{3}{2}A$ to him: Because in the one there are two arrangements where a is followed by b, & in the other there are two arrangements where b is followed by a; & partitioning each of the two other columns of six arrangements into three others of two arrangements, there will be two of these three which will give two times zero to the Banker, a & b occupying the second place, & the third will give to the Banker the same lot as he would have, if the card of the Punter would be found twice in the two cards; & consequently one would have again, according to this idea, the lot of the Banker,

$$= \frac{2 \times \overline{4 \times 2A + 2 \times \frac{3}{2}A} + 2 \times 2 \times 2A}{24} = A + \frac{1}{4}A.$$

One will notice again that the card of the Punter expressed by the letters a, b, c, being three times in the four cards, the three columns which begin with the letters a, b, c, will contain each two arrangements which will give $2A$ to the Banker, & four arrangements which will give to him $\frac{3}{2}A$, any two of the three letters a, b, c, being in sequence, & when partitioning the last column which commences with d into three columns of two arrangements, each of the three will give twice zero to the Banker, so that his lot will be again

$$\frac{3 \times \overline{2 \times 2A + 4 \times \frac{3}{2}A} + 1 \times 0}{24} = A + \frac{1}{4}A.$$

Finally it is evident that the card of the Punter expressed by the letters a, b, c, d, being four times in the four cards, the four columns which commence with the letters a, b, c, d, will contain each six arrangements, which will give to the Banker $\frac{3}{2}A$, since all these different arrangements will necessarily produce a doublet; whence it follows that the lot of the Banker will be $A + \frac{1}{2}A$.

All this is founded on the order of the arrangements, & it will be clarified by the application that I will make of it in the following.

Corollary III.

Whatever number of cards that the Banker holds, if that of the Punter is found there only one time, the advantage of the Banker will be expressed by a fraction which will have unity for the numerator, & for denominator the number of cards that the Banker holds: because six cards, for example, are able to be arranged in 720 different ways, it is clear that if one imagines all these different arrangements put on six columns of one hundred twenty arrangements each, in a way that in the first the letter a is everywhere in the first place, that in the second it is everywhere in the second place, that in the third it is everywhere in the third place, and thus in sequence, the first, the third & the fifth columns will give $2A$ to the Banker in all their arrangements; the second & the fourth will give to him zero, & the sixth will give to him A. One will have thus

$$s = \frac{3 \times 120 \times 2A + 2 \times 120 \times 0 + 1 \times 120 \times A}{720} = \frac{840}{720}A = A + \frac{1}{6}A.$$

And generally, if one names p the number of cards of the Banker; m the number of all the possible arrangements of these cards, one will have always the lot of the Banker expressed by this formula

$$s = \frac{\frac{1}{2}p \times \frac{m}{p} \times 2A + \frac{m}{p} \times A}{m} = A + \frac{A}{p}.$$

PROPOSITION III.

Second Case

One supposes that the Banker holds six cards, & that that of the Punter is in them a certain number of times. One requires what is the lot of the Banker in all the variations of this second case.

(1) Let be supposed that the card of the Punter is found twice in the six cards.

If these six cards are represented by the six letters a, b, c, d, e, f, g, in such a way that any two, for example, a & g express that of the Punter.

One will remark, (1) that one is able to put the seven hundred twenty different arrangements that six cards are able to receive on six columns, of which each will be composed of one hundred twenty perpendicular ranks; in such a way that the first column begins all with the letter a, the second with the letter b, the third with the letter c, & thus in sequence.

(2) That the two columns which begin with a & with g, are each ninety-six perpendicular ranks, which give to the Banker $2A$, and twenty-four which give to him $\frac{3}{2}A$: Because each rank of these two columns gives $2A$ to the Banker with the exception of those two where a is followed by g in the first, & where g is followed by a in the last. Now five letters are able to receive 120 different arrangements, & each being found necessarily an equal number of times after a in the first column, & after g in the last, it is evident that it is necessary to divide 120 by 4, in order to have all the doublets in each of the two columns which commence either with a, or with g. This remark is important for the solution of this Problem, & it is necessary to be remembered of it in the following.

The greatest difficulty, this is to discover that which the four other columns give to the Banker. In order to disentangle it, it is necessary to remark first that each of these four columns give an equal lot to the Banker (that which is evident,) & that thus it suffices to examine one of them. Let the column which commences with b, be the one which one wants to examine; & for greater facility, I partition it into five columns of twenty-four arrangements each.

1	2	3	4	5
$bacdfg$	$bcadfg$	$bdacfg$	$bfacdg$	$bgacdf$
$bacdgf$	$bcadgf$	$bdacgf$	$bfacgd$	$bgacfd$
$bacfdg$	$bcafdg$	$bdafcg$	$fbadcg$	$bgadfc$
$bacgdf$	$bcagfd$	$bdagcf$	$bfagcd$	$bgafcd$
$bacgfd$	$bcagdf$	$bdagfc$	$bfagdc$	$bgafdc$
$badcfg$	$bcdafg$	$bdcafg$	$bfcadg$	$bgcadf$
$badcgf$	$bcdagf$	$bdcagf$	$bfcagd$	$bgcafd$
$badfcg$	$bcdfag$	$bdcfag$	$bfcdag$	$bgcdaf$
$badfgc$	$bcdfga$	$bdcfga$	$bfcdga$	$bgcdfa$
$badgfc$	$bcdgaf$	$bdcgaf$	$bfcgad$	$bgcfad$
$badgcf$	$bcdgfa$	$bdcgfa$	$bfcgda$	$bgcfda$
$bafcdg$	$bcfadg$	$bdfacg$	$bfdacg$	$bgdacf$
$bafcgd$	$bcfagd$	$bdfagc$	$bfdagc$	$bgdafc$
$bafdcg$	$bcfdag$	$bdfcag$	$bfdcag$	$bgdcaf$
$bafdgc$	$bcfdga$	$bdfcga$	$bfdcga$	$bgdcfa$
$bafgdc$	$bcfgad$	$bdfgac$	$bfdgac$	$bgdfac$
$bafgcd$	$bcfgda$	$bdfgca$	$bfdgca$	$bgdfca$
$bagcdf$	$bcgadf$	$bdgacf$	$bfgacd$	$bgfacd$
$bagcfd$	$bcgafd$	$bdgafc$	$bfgadc$	$bgfadc$
$bagdcf$	$bcgdaf$	$bdgcaf$	$bfgcad$	$bgfcad$
$bagdfc$	$bcgdfa$	$bdgcfa$	$bfgcda$	$bgfcda$
$bagfcd$	$bcgfcd$	$bdgfac$	$bfgdac$	$bgfdac$
$bagfdc$	$bcgfda$	$bdgfca$	$bfgdca$	$bgfdca$

It is easy to see, in consulting this Table, that the first and the fifth columns give zero to the Banker, since in the first the letter a, & in the fifth the letter g hold the second place there, & that each of the three other columns contain twelve arrangements which give $2A$ to the Banker, eight which give to him zero, & four which give to him $\frac{3}{2}A$, that is to say, that each of these three columns give the same chances that one has found for the Banker in the preceding case, when one has supposed that he would hold four cards, among which that of the Punter would be found twice, of which the reason is that the first two letters of the second, third & fourth columns of the Table above being not that of the Punter, there remains four letters, among which that which expresses the card of the Punter are found twice: that which is reduced manifestly to the second article of the preceding case, where the card of the Punter is found twice in four cards.

Thus the column which begins with the letter b will give to the Banker $2 \times 24 \times 0 + 3 \times \overline{8 \times 0 + 12 \times 2A + 4 \times \frac{3}{2}A} = 90A$. Now the columns of 120 arrangements which begin with c, with d, & with f, give the same value, & consequently in order to have all the favorable coups that the four columns give which begin neither with a, nor with g, it is necessary to multiply $90A$ by 4, that which makes $360A$; to which adding $2 \times 96 \times 2A + 14 \times \frac{3}{2}A = 456A$ for the favorable coups that the columns give which begin with a & with g, one will have $\frac{360A+456A}{720} = \frac{816}{720}A = A + \frac{2}{15}A$ for the lot of the Banker in the proposed case.

COROLLARY

Whatever number of cards that the Banker holds, if that of the Punter is encountered there twice, in order to find the lot of the Banker, it is necessary to imagine all the possible arrangements of the cards which he holds set as many columns as there are cards; & to note next that the two columns which begin with the letters which express the card of the Punter, give each $2A$ to the Banker, with the exception of the ranks, where one of the letters which expresses the card of the Punter is followed by the other, which arrangement gives $\frac{3}{2}A$.

In order to find how many there are of those ranks in each of the two columns, it is necessary to divide all the arrangements which compose them by the number of the cards less one; the exponent of this division will express the number of the arrangements which give $\frac{3}{2}A$ in each of the two columns. In order to determine that which the other columns give, one will imagine them each partitioned into as many columns less one as there are cards; & observing an arrangement parallel to that of the two preceding Tables, one will find that there are always two of these last columns which give zero to the Banker, the two letters which express the card of the Punter occupying the second place there; & that each of the others equal to these will give to the Banker the same lot as he had in the preceding case, that is to say in the case where the number of the cards of the Banker being less by two, that of the Punter were twice there.

There is in the remarks of this Corollary what consists the solution of the Problem, for the case where the card of the Punter is found twice among the cards of the Banker. I would have had difficulty to well make understood this method, without making application of it in some particular cases, & without my making use of the Table which is found, page 10.

(2) In order to find what is the lot of the Banker, when he holds six cards among which that of the Punter is found three times, one will observe first that of six columns which express all the possible arrangements of these six cards, the three columns, of which the first letter expresses the card of the Punter, have each seventy-two arrangements which give $2A$ to the Banker, & forty-eight which give to him $\frac{3}{2}A$: For the three letters which express the card of the Punter being, for example, a, f, g, there are in the column which begins with the letter a twenty-four arrangements, where a is followed by f, & again twenty-four, where a is followed by g; it is likewise of the columns which begin with f & g.

In order to understand that which the three columns give, one will take care that partitioning one of the three columns of one hundred twenty perpendicular ranks into five others of twenty-four each, for example, the column which begins with b, thus as it is represented on page 10, there are three of these five which give zero to the Banker, namely those where the letters a, f & g are in the second place, & that each of the two other columns contain twelve arrangements which give $\frac{3}{2}A$ to the Banker, six which give to him zero, & six which give to him $2A$; that is that each of these two columns give the same chances as one has found for the Banker in the preceding case, when one has supposed that he held four cards, among which that of the Punter was found three times; of which the reason is that the first two letters of the second & of the third column of the Table, pag. 10, were not at all that of the Punter, there remains four letters among which that which expresses the card of the Punter is there three times, that which is reduced manifestly to the third article of the preceding case.

This put, the columns which begin with the letters b, c, d, will give

$$3 \times \overline{2 \times \overline{12 \times \frac{3}{2}A + 6 \times 2A}} = 180A;$$

to which adding

$$3 \times \overline{72 \times 2A + 48 \times \frac{3}{2}A} = 648A,$$

for this that the three columns which begin with the letters a, f, g give, one will have $\frac{828}{720}A = A + \frac{3}{20}A$ for the value which expresses the lot of the Banker in the case proposed.

COROLLARY

Whatever number of cards that the Banker holds, if that of the Punter is encountered three times, in order to find the lot of the Banker, it is necessary to imagine all the possible arrangements of the cards that the Banker holds set over as many columns as there are cards, & to note next that the three columns which begin with the letters which express the card of the Punter, give each $2A$ to the Banker, with the exception of the arrangements, where any two of the three letters which the card of the Punter is found next in the first & in the second place; these last give to him $\frac{3}{2}A$.

In order to find how many of these arrangements there are in each of the three columns, it is necessary to divide all the arrangements which compose them by the number of the cards less one: The exponent of this division multiplied by two will express the number of the arrangements which give $\frac{3}{2}A$ in each of these three columns.

In order to determine that which each of the three other columns give, one will imagine them each partitioned into as many columns less one, as there are cards; & observing an arrangement parallel to the one of the two preceding Tables, one will find that there are always three of these last columns which give zero to the Banker, the three letters which express the card of the Punter occupying the second place; & that each of the others equal to the latter will give to the Banker the same lot as he had in the preceding case, where, the number of cards of the Banker being less by three, that of the Punter was twice.

(3) In order to find what is the lot of the Banker, when he holds six cards, among which that of the Punter is found four times.

One will note that expressing as above the six cards by the letters a, b, c, d, f, g, of which any four, for example, a, d, f, g, designating that of the Punter, if one distributes the seven hundred twenty possible arrangements of these six cards on six columns, of which the first begins always with the letter a, the second with the letter b, &c. as it has been said above, the four columns, of which the first letter expresses the card of the Punter, would contain each 48 arrangements which give $2A$ to the Banker, & seventy-two which give to him $\frac{3}{2}A$. For of the four letters which express the card of the Punter, there are three which follow the letter a in the first column, & it is likewise of the other columns which begin with the letters d, f, g.

In order to understand that which the two other columns give, it is necessary to observe that partitioning one of these two columns of one hundred twenty perpendicular ranks into five others of twenty-four each, for example, the column which begins with b, thus as it is represented in the Table page 10; there are four of these five which give zero to the Banker, namely, those where the letters a, d, f, g, are in the second place, & that the other contains twenty-four arrangements, which give to the Banker $\frac{3}{2}a$; the reason is evident.

This supposed, the columns which begin with the letters a, d, f, g, will give all together $4 \times \overline{48 \times 2A + 72 \times \frac{3}{2}A}$, to which adding $+2 \times \overline{1 \times 24 \times \frac{3}{2}A}$, for this that the two columns which begin with the letters b & c give, one will have $\frac{888}{720}A = A + \frac{7}{30}A$, for the value which expresses the lot of the Banker in the proposed case.

COROLLARY.

Whatever number of cards the Banker holds, if that of the Punter is encountered four times, in order to find the lot of the Banker, it is necessary to imagine all the possible arrangements of the cards which he holds put on as many columns as there are cards, & to note that the four columns which begin with the letters which express the card of the Punter, give each $2A$ to the Banker, with the exception of the ranks, where any two of the four letters which express the card of the Punter is found consecutively in the first & in the second place; these last give to him $\frac{3}{2}A$.

In order to find how many arrangements there are in each of the four columns, it is necessary to divide all the arrangements which compose them by the number of cards less one. The exponent of this division multiplied by three will express the number of arrangements which give $\frac{3}{2}A$ in each of these four columns.

In order to determine that which each of the two other columns gives, one will imagine them partitioned into as many columns less one as there are cards; & observing an arrangement similar to the one of the two Tables pag. 5 & 10, one will find that there are always four of these last columns which give zero to the Banker, the four letters which express the card of the Punter occupying the second place, & that each of the others equal to the former will give to the Banker the same lot that one has found for the Banker in the case, where the number of cards of the Banker being less by two, that of the Punter was four times.

PROPOSITION III.

THIRD CASE.

One supposes that there remain eight cards in the hand of the Banker, & that in these eight cards that of the Punter is two times; one demands what is in this case the advantage of the Banker.

(1) Let the eight cards of the Banker be represented by the eight letters a, b, c, d, f, g, h, k; let also that of the Punter be designated by any two a & b.

One knows by that which had been said in the solution of the first case page 5, that eight cards are able to be arranged in forty thousand three hundred twenty different ways: If therefore one places all these different arrangements on eight columns, which each contain five thousand forty of them, so that all the arrangements of the first column begin with the letter a, all the arrangements of the second with the letter b, & thus consecutively.

One will note that the first two columns each contain four thousand three hundred twenty arrangements which give $2A$ to the Banker, & seven hundred twenty arrangements which give to him $\frac{3}{2}A$.

In order to determine that which each of the six other columns gives, one will subdivide them into seven others each composed of seven hundred twenty arrangements according to the order of the Table pag. 10, & one will observe that of these seven columns there are two which give zero, namely those where a & b are in the second place, & that the five others each contain two hundred eighty-eight arrangements which give zero to the Banker, three hundred thirty-six which give to him $2A$, & ninety-six which give to him $\frac{3}{2}A$; that is that each of these five columns give to the Banker the same lot as one has found, when in the six cards that of the Punter was found twice: Whence it is necessary to conclude that each of the six columns, which begin neither with a nor with b, will give to the Banker $4080A$, & consequently the six together will give $24480A$. Adding therefore to this value that which the two columns give of which the arrangements begin with a & with b, one will have $\frac{24480A+19440A}{40320} = \frac{4392}{4032}A = A + \frac{5}{56}A$ for the value which expresses the lot of the Banker in the proposed case.

(2) One will find in the same manner that the lot of the Banker, when in his eight cards that of the Punter is found three times, is $= \frac{3\times3600\times2A+1440\times\frac{3}{2}A}{40320} + \frac{5\times828\times4\times A}{40320} = \frac{44640}{40320}A = A + \frac{3}{28}A$.

(3) One will find similarly that the lot of the Banker, when in his eight cards that of the Punter is found four times, is $= \frac{4\times2880\times2A+2160\times\frac{3}{2}A}{40320} + \frac{4\times888\times3A}{40320} = \frac{46656}{40320}A = A + \frac{11}{70}A$.

GENERALLY.

Whatever number of cards that the Banker holds, & whatever number of times that the card of the Punter be among those of the Banker, one will find always his lot in this way. (1) One will seek by the method 5, the number of all the different possible arrangements of the cards of the Banker. (2) One will represent these cards by the letters a, b, c, d, f, &c. & one will suppose that some at will designate that of the Punter. (3) One will imagine all these different arrangements distributed on as many columns as there will be cards; in such a way that the first begins all with the letter a, the second with the letter b, the third with the letter c, &c. (4) One will note that the columns which begin with the letters which designate the card of the Punter, give $2A$ to the Banker in all their arrangements, with the exception of those where some two of among the letters which express the card of the Punter, are found in sequence in the first & in the second place; this will give $\frac{3}{2}A$.

In order to find the number of those arrangements in each of these columns, one will divide the number of arrangements from which each column is composed by the number of the cards of the Banker less one, & one will multiply the exponent by the number of times less one that the card of the Punter is found in those of the Banker; this product will give all the arrangements of these columns, which give $\frac{3}{2}A$.

In regard to the other columns which begin with some letters different from those which express the card of the Punter, it is necessary, in order to discover the favorable arrangements there, to imagine them each partitioned & subdivided into as many columns less one, as there are cards, & to have regard to the order marked in the Tables on pages 5 and 10; to observe that of those last columns there are always as many which give zero to the Banker, as of times the card of the Punter is found in those of the Banker; & that each of the other small columns give to the Banker the same lot as one has found in the case which has preceded; that is to say in the case where the number of cards of the Banker being less by two, the card of the Punter is found there an equal number of times.

Thus one will find among all the different possible arrangements of the cards which the Banker holds, which are those which give to him either A, or $2A$, or $\frac{3}{2}A$, or zero; consequently one will have by this method the lot of the Banker in all the possible cases: that which is was necessary to find.

In following the idea of this demonstration, if one names p the number of cards that the Banker holds, q the number of times that the card of the Punter is in those of the Banker, g the lot of the Banker in a number of cards expressed by $p-2$, S the sought lot: one will have the lot of the Banker expressed by this formula.

$$S = \frac{\overline{pq-qq} \times 2A + \overline{qq-q} \times \frac{3}{2}A + g \times \overline{p-q} \times \overline{p-q-1}}{p \times \overline{p-1}}$$

One is able to find by this formula the lot of the Banker, whatever number of cards that he have within the hands, & whatever number of times that the card of the Punter is contained there. But this formula has this very great inconvenience of giving the advantage of the Banker for a certain number of cards designated by p, only when one knows already his advantage for a number of cards which is $p-2$. Thus this formula is only able to be useful in order to find all the different cases the one after the other, by beginning with the most simple. Here is another much more refined, & infinitely more extensive, which gives without much calculation all the different cases in general, & each case in particular independently from one another.

The advantage of the Banker is able to be expressed by a fraction which one will determine by the rule which follows. The denominator will contain as many products of the quantities p, $p-1$, $p-2$, &c. as there are units in q. In order to have the numerator it will be necessary to take in the Table of combinations page 54, a horizontal rank, of which the quantity is $q-1$, to add into one sum all the terms of this rank which are in odd number, to begin with the one which corresponds to $p-2$ in the rank of the natural numbers, that is the first, the third, the fifth, &c. to zero; to multiply this sum by as many products of the natural numbers 1, 2, 3, 4, 5, 6, &c. as there are units in q, & to multiply again by $\frac{1}{2}A$, with the sole exception of q being 2, it is necessary to multiply the last term of the series by A, & not by $\frac{1}{2}A$. This rule will be clarified by an Example.

Let $q = 6$, & $p = 14$, it is necessary to add the terms of the fifth horizontal band, to begin with 495 which corresponds to $12 = p-2$, these terms are $495 + 210 + 70 + 15 + 1$, of which the sum being multiplied by 1, 2, 3, 4, 5, 6, &c. multiplied again by $\frac{1}{2}A$, & divided by 14, 13, 12, 11, 10, 9, is $\frac{791}{6006}A$.

In order to reduce this rule to some formulas which determine all at once, by substituting for p its value, the advantage of the Banker for whatever number of cards as it be, when the card of the Punter is found a certain number of times expressed by q; it will suffice in the case of $q = 3$, to find the sum of an arithmetic progression; thus the first three cases where $q = 1 = 2 = 3$, have no difficulty. In regard to the others, one has need of the general solution of the Problem which follows: *To find the sum of a progression of which each term is formed of as many products of quantities which decrease by unity, as there are units in* $q-2$.

Thus q being 4, the concern is to find the sum of a progression of which the first term is $p-2 \times p-3 + p-4 \times p-5 + p-6 \times p-7$ &c. If $q = 5$, it is necessary to find the sum of this progression $p-2 \times p-3 \times p-4 + p-4 \times p-5 \times p-6 \times p-7 \times p-8+$&c. If $q = 6$, it is necessary to find the sum of this progression $p-2 \times p-3 \times p-4 \times p-5 + p-4 \times p-5 \times \times p-6 \times p-7 + p-6 \times p-7 \times p-8 \times p-9+$&c. & thus consecutively with respect to the different values of q.

As the method which serves to find the sum of these series is rather singular, & is able to appear to have some difficulty, I will have wished to explicate it here; but as it supposes that which is demonstrated from page 54 to page 100, I will content myself by giving the following formulas, $B \overset{1}{=} \frac{A}{p}$, $C \overset{2}{=} \frac{p+2}{2 \times \overline{pp-p}}A$, $D \overset{3}{=} \frac{3}{4 \times \overline{p-1}}A$, $E \overset{4}{=} \frac{2p-5}{2pp-8p+6}A$, $F \overset{5}{=} \frac{5p-10}{4 \times \overline{pp-4p+3}}A$, $G \overset{6}{=} \frac{2pp-13p+16}{p^3-9pp+23p-15} \times \frac{3}{4}A$, $H \overset{7}{=} \frac{7}{8} \times \frac{2pp-12p+13}{p^3-9pp+23p-15}A$, $K \overset{8}{=} \frac{4p^3-50pp+176p-151}{p^4-16p^3+86pp-176p+105} \times \frac{1}{2}A$, $L = \overset{9}{=}$, &c. to observe a property quite singular of the figurate numbers, which has served me as foundation in order to find the sums of the preceding series. This property consists in this that ranking these numbers in triangle in the manner of page 54, if one names any natural even number p, the sum of the numbers of any horizontal rank which corresponds to some odd natural numbers, will be equal to the excess of the numbers which in the following rank correspond to some even natural numbers, by beginning with p on the numbers of this last rank which correspond to some odd natural numbers.

The first of these formulas expresses the advantage of the Banker when the card of the Punter is found once in his hand, the second expresses his advantage when it is found twice, the third expresses his advantage when it is found three times, & thus consecutively.

I have prepared two Tables on the first four formulas, in the intent to please the Players, & to satisfy their curiosity. In order to understand the usage, it is necessary to know that in the first the number contained within the cell $\square$ expresses the number of cards that the Banker holds; & that the number which follows, either the cell in the first column, or two points in the other columns, expresses the number of times that the card of the Punter is supposed to be found in the hand of the Banker. The usage of the second Table is to give some expressions less exact to the truth, but more simple & more intelligible to the players, of the fractions which in the first designate with precision the advantage of the Banker. It is necessary to know in order to understand this Table, that this mark $>$ signifies excess, & this other $<$defect; in such a way that I intend by $> \frac{1}{4} < \frac{1}{3}$ a quantity greater than $\frac{1}{4}$, & smaller than $\frac{1}{3}$.

One is able to make, by relationship to the numbers of the first Table, many rather curious observations. Here is the most important.

COROLLARY I.

In the first Table the advantage of the Banker is expressed in the first column by a fraction of which the numerator is always unity, the denominator is the number of cards which the Banker holds.

In the second column this advantage is expressed by a fraction, of which the numerator being according to the sequence of natural numbers $1, 2, 3, 4$, &c. the denominator has for difference among its terms the numbers $18,\ 26, 34, 42,\ 50,\ 58,$ of which the difference is 8.

In the third column the numerator being always 3, the difference which rules in the denominator is 8.

In the fourth column the difference being always 4 in the numerator, the denominator has for difference among its terms the numbers $24,\ 40,\ 56,\ 72,\ 88,$ &c. of which the difference is 16.

One is able again to observe another uniformity rather singular among the last digits of the denominator of each term of a column.

In the first column the last digits of the denominator are according to this order $4, 6, 8, 0,\ 2|4, 6, 8, 0, 2|$&c. They are according to this order $2, 0, 6, 0, 2|2, 0, 6, 0, 2|$ &c. in the second. In the third they are according to this order $2, 0, 8, 6, 4|2, 0, 8, 6, 4|$ &c. In the fourth they are according to this order $6, 0, 0, 6, 8|6, 0, 0, 6, 8|$ &c. One will seek with pleasure the cause of this uniformity.

COROLLARY II.

One will be able by means of these Tables to find all in one stroke how much a Banker has in advantage on each card. One may likewise know how much each complete deal will have, in equal fortune, to bring profit to the Banker, if one remembers the number of cards which have been taken by the Punters, the diverse circumstances in which one has wagered on them in the game, & finally the amount of money that one has ventured upon. One will find apparently that this advantage is very considerable. One would give to him fair limits in establishing that the doublets were indifferent for the Banker & for the Punter, or at least that they are worth solely the third or the fourth of the stake of the Punter. Thus that which would remain of advantage to the Banker, would be sufficient for making preference to the players who understand their interest, the place of the Banker to that of the Punter, & it would not be considerable enough, in order that the Punters would suffer by them much prejudice from them.

COROLLALRY III.

So that the Punter taking a card has the least disadvantage which is possible, it is necessary that he choose one which has passed twice; because there would be greater disadvantage for him, if he took a card which has passed once; & greater disadvantage again, if he took a card which has passed three times; & finally the worst choice that a Punter is able to make, this is to take a card which has not passed yet.

Thus one will find, for example, that supposing A is equal to one pistole, the advantage of the Banker which would be nineteen sols two deniers, under the supposition that the card of the Punter was four times in twelve cards; & sixteen sols eight deniers, under the supposition that it was there once, is no more than thirteen sols seven deniers, when in these twelve cards that of the Punter is found there three times, & ten sols seven deniers when it is there only twice.

One will note the same thing with respect to each other number of cards.

REMARK I.

The persons who have not examined at foundation the game of Pharaon & of Bassete, could find to criticize, that I have not spoken of the masses, of the parolis, of the paix, of the sept & the *va*, &c. because the majority of the players imagine that there is in everything that much mystery. I have known of them who believed to have good reasons to prefer to wager four Louis on a single card to make the paroli of two Louis, or the sept & the *va* of one Louis. I have seen others of them who were persuaded that there would be much advantage to make frequently the paix: nevertheless it is evident that, since the Punter has the liberty to take a new card at each time that he loses or that he wins such as it pleases him, he must not concern himself if this is either a sept & the *va*, or a paroli, or a paix, or a double paix, &c. Because to make the paroli of a Louis is nothing other than to set two Louis on one card, after having won one Louis; & to make the sept & the *va* of a Louis is nothing other than to set four Louis on one card, after having won three of them; & similarly to make the paix of one Louis is nothing other than to set a Louis on one card, after having won one Louis on that same card.

One has apparently invented the parolis, the sept & the *va*, &c. only in order to spare the Banker the difficulty of paying those who have intent to set on their cards the double of that which they have just won: nonetheless it would be more useful to the Bankers to take this care, than to be exposed, as they are, to that which one names Alpiou de Campagne.

For me I believe that if the Bankers have not abolished the usage making these points, of which the great number cause in the Game a confusion which is often prejudicial to the Banker, & and which favors the misdirections of the Punters, it is that the Bankers have well seen that the majority of men do not judge some things by reason, such a Punter who would make without difficulty the sept & the *va* of one Louis, believing to risk only one Louis, could not be able to be resolved to set four Louis on a single card. Besides for the usual it is in the last cards, when the advantage of the Banker is most considerable, that the Punters are stung & make the parolis, the sept & the *va*, &c. that which shall compensate them with usage of the misdirections to which they are that way exposed, but of which it is not moreover impossible to guarantee themselves with much application, & with the aid of a croupier.

REMARK II.

It was easy for the players themselves to understand that the advantage of the Banker increases in proportion as the number of his cards diminish; but it was impossible to discover without Analysis the law of this diminution, & that which is most important, to know how this advantage varies according as the card of the Punter is found more or less times in the hand of the Banker. The players have assuredly never been able to imagine that the advantage of the Banker, in relation to one card which has not passed, is nearly double of that which he has on one card which has passed twice, & much less again than his advantage, in relation to one card which has passed three times, is to his advantage in relation to one card which has passed two times in a ratio greater than three to two. The players will find all this without difficulty, & possibly with some surprise in the Tables here joined, they will see there, for example, that the advantage of the Banker which would only be about twenty-four sols if the Punter would set six pistoles either to the first deal of the game, or on one card which would have passed twice when there remained no more than twenty-eight of them in the hand of the Banker (these two cases revert to very nearly the same thing) will be seven livres two sols, if the Punter sets six pistoles on one card which has not yet passed, the stock being composed of no more than ten cards, & that his advantage would be precisely 6 livres, if the card of the Punter was in this last case passed three times. Thus all the knowledge of the game is reduced for the Punter to observing the two rules which follow.

(1) Take some cards only in the first deals, & venture on the game accordingly less as there are a greater number of deals passed.

(2) Regard as the greatest evils those cards which have not passed at all yet, or which have passed three times, & prefer to all, those which have passed twice.

In following these two rules, the disadvantage of the Punter will be the least that will be possible.

TABLE FOR PHARAON

52	$1 = * \quad * \quad *$	$: 2 = * \quad * \quad *$	$: 3 = * \quad * \quad *$	$: 4 = a + \frac{99}{4998}a$
50	$1 = * \quad * \quad *$	$: 2 = a + \frac{26}{2450}a$	$: 3 = a + \frac{3}{196}a$	$: 4 = a + \frac{95}{4606}a$
48	$1 = a + \frac{1}{48}a$	$: 2 = a + \frac{25}{2256}a$	$: 3 = a + \frac{3}{188}a$	$: 4 = a + \frac{91}{4230}a$
46	$1 = a + \frac{1}{46}a$	$: 2 = a + \frac{24}{2070}a$	$: 3 = a + \frac{3}{180}a$	$: 4 = a + \frac{87}{3870}a$
44	$1 = a + \frac{1}{44}a$	$: 2 = a + \frac{23}{1892}a$	$: 3 = a + \frac{3}{172}a$	$: 4 = a + \frac{83}{3526}a$
42	$1 = a + \frac{1}{42}a$	$: 2 = a + \frac{22}{1722}a$	$: 3 = a + \frac{3}{164}a$	$: 4 = a + \frac{79}{3198}a$
40	$1 = a + \frac{1}{40}a$	$: 2 = a + \frac{21}{1560}a$	$: 3 = a + \frac{3}{156}a$	$: 4 = a + \frac{75}{2886}a$
38	$1 = a + \frac{1}{38}a$	$: 2 = a + \frac{20}{1406}a$	$: 3 = a + \frac{3}{148}a$	$: 4 = a + \frac{71}{2590}a$
36	$1 = a + \frac{1}{36}a$	$: 2 = a + \frac{19}{1260}a$	$: 3 = a + \frac{3}{140}a$	$: 4 = a + \frac{67}{2310}a$
34	$1 = a + \frac{1}{34}a$	$: 2 = a + \frac{18}{1122}a$	$: 3 = a + \frac{3}{132}a$	$: 4 = a + \frac{63}{2046}a$
32	$1 = a + \frac{1}{32}a$	$: 2 = a + \frac{17}{992}a$	$: 3 = a + \frac{3}{124}a$	$: 4 = a + \frac{59}{1798}a$
30	$1 = a + \frac{1}{30}a$	$: 2 = a + \frac{16}{870}a$	$: 3 = a + \frac{3}{116}a$	$: 4 = a + \frac{55}{1566}a$
28	$1 = a + \frac{1}{28}a$	$: 2 = a + \frac{15}{756}a$	$: 3 = a + \frac{3}{108}a$	$: 4 = a + \frac{51}{1350}a$
26	$1 = a + \frac{1}{26}a$	$: 2 = a + \frac{14}{650}a$	$: 3 = a + \frac{3}{100}a$	$: 4 = a + \frac{47}{1150}a$
24	$1 = a + \frac{1}{24}a$	$: 2 = a + \frac{13}{552}a$	$: 3 = a + \frac{3}{92}a$	$: 4 = a + \frac{43}{966}a$
22	$1 = a + \frac{1}{22}a$	$: 2 = a + \frac{12}{462}a$	$: 3 = a + \frac{3}{84}a$	$: 4 = a + \frac{39}{798}a$
20	$1 = a + \frac{1}{20}a$	$: 2 = a + \frac{11}{380}a$	$: 3 = a + \frac{3}{76}a$	$: 4 = a + \frac{35}{646}a$
18	$1 = a + \frac{1}{18}a$	$: 2 = a + \frac{10}{306}a$	$: 3 = a + \frac{3}{68}a$	$: 4 = a + \frac{31}{510}a$
16	$1 = a + \frac{1}{16}a$	$: 2 = a + \frac{9}{240}a$	$: 3 = a + \frac{3}{60}a$	$: 4 = a + \frac{27}{390}a$
14	$1 = a + \frac{1}{14}a$	$: 2 = a + \frac{8}{182}a$	$: 3 = a + \frac{3}{52}a$	$: 4 = a + \frac{23}{286}a$
12	$1 = a + \frac{1}{12}a$	$: 2 = a + \frac{7}{132}a$	$: 3 = a + \frac{3}{44}a$	$: 4 = a + \frac{19}{198}a$
10	$1 = a + \frac{1}{10}a$	$: 2 = a + \frac{6}{90}a$	$: 3 = a + \frac{3}{36}a$	$: 4 = a + \frac{15}{126}a$
8	$1 = a + \frac{1}{8}a$	$: 2 = a + \frac{5}{56}a$	$: 3 = a + \frac{3}{28}a$	$: 4 = a + \frac{11}{70}a$
6	$1 = a + \frac{1}{6}a$	$: 2 = a + \frac{4}{30}a$	$: 3 = a + \frac{3}{20}a$	$: 4 = a + \frac{7}{30}a$
4	$1 = a + \frac{1}{4}a$	$: 2 = a + \frac{3}{12}a$	$: 3 = a + \frac{3}{12}a$	$: 4 = a + \frac{3}{6}a$

TABLE II. FOR PHARAON

52	$1 = * \quad * \quad *$	$: 2 = * \quad * \quad *$	$: 3 = * \quad * \quad *$	$: 4 = a+ > \frac{1}{51} < \frac{1}{50}$
50	$1 = * \quad * \quad *$	$: 2 = a+ > \frac{1}{95} < \frac{1}{94}$	$: 3 = a+ > \frac{1}{66} < \frac{1}{65}$	$: 4 = a+ > \frac{1}{49} < \frac{1}{48}$
48	$1 = a + \frac{1}{48}a$	$: 2 = a+ > \frac{1}{91} < \frac{1}{90}$	$: 3 = a+ > \frac{1}{63} < \frac{1}{62}$	$: 4 = a+ > \frac{1}{47} < \frac{1}{46}$
46	$1 = a + \frac{1}{46}a$	$: 2 = a+ > \frac{1}{87} < \frac{1}{86}$	$: 3 = a + \frac{1}{60}$	$: 4 = a+ > \frac{1}{45} < \frac{1}{44}$
44	$1 = a + \frac{1}{44}a$	$: 2 = a+ > \frac{1}{83} < \frac{1}{82}$	$: 3 = a+ > \frac{1}{58} < \frac{1}{57}$	$: 4 = a+ > \frac{1}{43} < \frac{1}{42}$
42	$1 = a + \frac{1}{42}a$	$: 2 = a+ > \frac{1}{79} < \frac{1}{78}$	$: 3 = a+ > \frac{1}{55} < \frac{1}{54}$	$: 4 = a+ > \frac{1}{41} < \frac{1}{40}$
40	$1 = a + \frac{1}{40}a$	$: 2 = a+ > \frac{1}{75} < \frac{1}{74}$	$: 3 = a + \frac{1}{52}$	$: 4 = a+ > \frac{1}{39} < \frac{1}{38}$
38	$1 = a + \frac{1}{38}a$	$: 2 = a+ > \frac{1}{71} < \frac{1}{70}$	$: 3 = a+ > \frac{1}{50} < \frac{1}{49}$	$: 4 = a+ > \frac{1}{37} < \frac{1}{36}$
36	$1 = a + \frac{1}{36}a$	$: 2 = a+ > \frac{1}{67} < \frac{1}{66}$	$: 3 = a+ > \frac{1}{47} < \frac{1}{46}$	$: 4 = a+ > \frac{1}{35} < \frac{1}{34}$
34	$1 = a + \frac{1}{34}a$	$: 2 = a+ > \frac{1}{63} < \frac{1}{62}$	$: 3 = a + \frac{1}{44}$	$: 4 = a+ > \frac{1}{33} < \frac{1}{32}$
32	$1 = a + \frac{1}{32}a$	$: 2 = a+ > \frac{1}{59} < \frac{1}{58}$	$: 3 = a+ > \frac{1}{42} < \frac{1}{41}$	$: 4 = a+ > \frac{1}{31} < \frac{1}{30}$
30	$1 = a + \frac{1}{30}a$	$: 2 = a+ > \frac{1}{55} < \frac{1}{54}$	$: 3 = a+ > \frac{1}{39} < \frac{1}{48}$	$: 4 = a+ > \frac{1}{29} < \frac{1}{28}$
28	$1 = a + \frac{1}{28}a$	$: 2 = a+ > \frac{1}{51} < \frac{1}{50}$	$: 3 = a + \frac{1}{36}$	$: 4 = a+ > \frac{1}{27} < \frac{1}{26}$
26	$1 = a + \frac{1}{26}a$	$: 2 = a+ > \frac{1}{47} < \frac{1}{46}$	$: 3 = a+ > \frac{1}{34} < \frac{1}{33}$	$: 4 = a+ > \frac{1}{25} < \frac{1}{24}$
24	$1 = a + \frac{1}{24}a$	$: 2 = a+ > \frac{1}{43} < \frac{1}{42}$	$: 3 = a+ > \frac{1}{31} < \frac{1}{30}$	$: 4 = a+ > \frac{1}{23} < \frac{1}{22}$
22	$1 = a + \frac{1}{22}a$	$: 2 = a+ > \frac{1}{39} < \frac{1}{38}$	$: 3 = a + \frac{1}{28}$	$: 4 = a+ > \frac{1}{21} < \frac{1}{20}$
20	$1 = a + \frac{1}{20}a$	$: 2 = a+ > \frac{1}{35} < \frac{1}{34}$	$: 3 = a+ > \frac{1}{26} < \frac{1}{25}$	$: 4 = a+ > \frac{1}{19} < \frac{1}{18}$
18	$1 = a + \frac{1}{18}a$	$: 2 = a+ > \frac{1}{31} < \frac{1}{30}$	$: 3 = a+ > \frac{1}{23} < \frac{1}{22}$	$: 4 = a+ > \frac{1}{17} < \frac{1}{16}$
16	$1 = a + \frac{1}{16}a$	$: 2 = a+ > \frac{1}{27} < \frac{1}{26}$	$: 3 = a + \frac{1}{20}$	$: 4 = a+ > \frac{1}{15} < \frac{1}{14}$
14	$1 = a + \frac{1}{14}a$	$: 2 = a+ > \frac{1}{23} < \frac{1}{22}$	$: 3 = a+ > \frac{1}{18} < \frac{1}{17}$	$: 4 = a+ > \frac{1}{13} < \frac{1}{12}$
12	$1 = a + \frac{1}{12}a$	$: 2 = a+ > \frac{1}{19} < \frac{1}{18}$	$: 3 = a+ > \frac{1}{15} < \frac{1}{14}$	$: 4 = a+ > \frac{1}{11} < \frac{1}{10}$
10	$1 = a + \frac{1}{10}a$	$: 2 = a + \frac{1}{15}$	$: 3 = a + \frac{1}{12}$	$: 4 = a+ > \frac{1}{9} < \frac{1}{8}$
8	$1 = a + \frac{1}{8}a$	$: 2 = a+ > \frac{1}{12} < \frac{1}{11}$	$: 3 = a+ > \frac{1}{10} < \frac{1}{9}$	$: 4 = a+ > \frac{1}{7} < \frac{1}{6}$
6	$1 = a + \frac{1}{6}a$	$: 2 = a+ > \frac{1}{8} < \frac{1}{7}$	$: 3 = a+ > \frac{1}{7} < \frac{1}{6}$	$: 4 = a+ > \frac{1}{5} < \frac{1}{4}$
4	$1 = a + \frac{1}{4}a$	$: 2 = a + \frac{1}{4}$	$: 3 = a + \frac{1}{4}a$	$: 4 = a + \frac{1}{2}a$

PROBLEM
ON THE GAME
OF LANSQUENET.

To determine generally the advantage of the one who has the hand, & the lot of the other players with respect to the different places that they occupy.

One names coupeurs those who take a card in the round, before the one who has the hand is given his; & the carabineurs, those who take a card, after that of the one who has the hand is drawn. One calls the réjouissance the card which comes immediately after the card of the one who has the hand. Everyone is able to set before the card of the one who has the hand is drawn; but it depends on him to keep that which he wishes, provided that it is explicated before drawing his card: for if he draws it without saying anything, he is obliged to keep all that which one has set.

After one has regulated the fund of the game, the one who has the hand deals some cards to the coupeurs by beginning with his right, & these cards are named right cards, in order to distinguish them from the cards of reprise and réjouissance; he gives himself a card, & next he draws the réjouissance. That being done, he continues to draw all the cards in sequence; he wins that which is on the card of the coupeur, when he brings forth the card of this coupeur; & he loses all that which is in the game, when he brings forth his. Finally if he brings forth all the right cards of the coupeurs before he brings forth his, he recommences & continues to have the hand, whether he has won or lost the réjouissance. Here are the most general rules of this Game: Here are some other particulars which have relation to the proposed Problem.

(1) When the one who has the hand, who I will name always Pierre, gives a double card to the coupeur, that is a card of same kind as another card that he has already given to another coupeur who is more to his right, he wins the fund of the game on the losing card, & he is obliged to keep the double on the double card.

(2) When Pierre gives a triple card to a coupeur, he wins that which is on the losing card, & he is held to set four times the fund of the game on the triple card.

(3) When Pierre gives a quadruple card to a coupeur, he retakes that which he has set on the single or double cards; if there is it, he loses that which is on the triple card of the same kind as the quadruple that he brings forth, & he quits the hand immediately, without giving other cards.

(4) If a quadruple card is given to him, he takes he takes all that which there is on the cards on the coupeurs, &, without giving other cards, he recommences the hand.

(5) When the card of the réjouissance is quadruple, he goes not at all.

(6) There is further a law of the game, that a coupeur, of whom the card is taken, is obliged to pay the fund of the Game to each coupeur who has a card before him, that which is called arroser: but there is this distinction to make, that when it is a right card, the one who loses pays to the other right cards the fund of the game, without having regard to this that his or the right card of the other players is single, double or triple; instead when it is a card of reprise, one pays & receives only according to the rules of the parts. Now in this Game the parts are set three against two, when one has a double card against single card;

two against one, when one has a triple card against double card, & three against one, when one has a triple card against single card.

These rules being well known, if one wishes to know in what consists the difficulty of the first part of this Problem, which is to determine the advantage of the one who has the hand, it is necessary to observe,

(1) That the advantage to have the hand contains in it another quite considerable, which is to conserve to Pierre the right to keep the cards as many times as he will have brought forth all the right cards of the coupeurs before bringing forth his. Now as that is able to happen many times consecutively, such number of coupeurs as there be, it is necessary, in examining the advantage of the one who holds the cards, to have regard to the expectation that he has to make the hand any number of times indeterminately. Whence it follows that one is able to express the advantage of Pierre only by a series composed of an infinite number of terms, which will always decrease; that which gives some subject to believe that one can never have the precise value of the advantage of Pierre, but only a value so much more exact, as one will employ a greater number of terms of the series.

(2) That Pierre has so much less expectation to make the hand, as there are more coupeurs & more single hands among the right cards.

(3) That the obligation where Pierre is to set the double of the fund of the game on the double cards, & the quadruple on the triple cards, diminishes the advantage that he would have in bringing forth the double or triple cards before being given his; & that his advantage is increased by this other condition of the game, which permits him to retake in whole that which he has set on some double & triple cards, when he gives to one of the coupeurs a quadruple card.

These remarks, & some similar others that I omit, is able to make known that this Problem is more composed than it appears at first.

In order to resolve it, here is the route that I take. I examine first all the different dispositions that the game is able to have, before Pierre himself is given his card, & I determine how much probability there be that each of the possible dispositions will be found to the exclusion of the others. Next I seek what is the expectation of Pierre in each of these different dispositions of being given a card either single, or double, or triple, or quadruple. In third place, I examine in particular that which each of the different relations of the card of Pierre to those of the coupeurs is able to give to him of gain or of loss. Finally after these researches, there remains only to operate according to the ordinary rules of Analysis. It would be long & difficult to make the method understood, without making application on some particular cases: thus, without extending myself further, I begin as in the preceding Problem with the singlest case.

PROPOSITION V.
First Case

Let one suppose that there are three coupeurs, Pierre, Paul & Jacques. Paul is the first to the right, & Jacques the second. One demands how much is the advantage for Pierre to have the hand.

Let the fund of the game be called A.

One will note:

(1) That there is odds sixteen against one, that the cards of Paul & of Jacques will be found single, when Pierre will be at the point to draw his card & one against sixteen that the card of Jacques will be found double.

(2) That the cards of Paul & of Jacques being single, Pierre has six coups out of fifty in order to bring forth double card, & consequently forty-four out of fifty, in order to bring forth single card.

(3) That the card of Jacques being double, Pierre has two coups out of fifty in order to win all, by bringing forth triple card, & consequently forty-eight out of fifty, in order to bring forth single card.

(4) That if Pierre brings forth single card, the cards of Paul & of Jacques being single, his lot is $2A$; that which is evident: but that if Pierre brings forth double card, his lot is $3A + \frac{1}{5}A$. Because bringing forth double card, he takes first $2A$, that is the stake of the one who loses & his own, & beyond that he has his stake on the card of the Player who remains, & the advantage of having double card against single card: now this advantage is $\frac{1}{5}A$; here is the proof. Pierre having double card against single card, has three coups in order to win, & only two coups in order to lose; his lot will be therefore in this case $3 \times 2A + 2 \times 0$ divided by 5. Therefore his lot will be $A + \frac{1}{5}A$, & his advantage $\frac{1}{5}A$.

(5) That if Pierre brings forth single card, the card of Jacques being double, his lot is $2A - \frac{2}{5}A$. Because it is a law of the game, that Jacques having double card is by right to set $2A$ on his card, & to oblige Pierre to set so much on it although to his disadvantage. One has seen above that the advantage to the one who has double card against single card is the fifth part of the stake of each: now here the stake of Pierre being $2A$, his advantage & the disadvantage of Jacques will be $\frac{2}{5}A$. It is evident that if Pierre would bring forth triple card, his lot would be $4A$.

(6) It is necessary to observe that Pierre risks $2A$, when the cards of Paul & Jacques are single; but that he risks only A, when the card of Jacques is double: From all that it follows that the advantage that Pierre has in a round is $\frac{16}{17} \times \frac{18}{125}A + \frac{1}{17} \times \frac{87}{125}A = \frac{375}{2125}A = \frac{3}{17}A$.

Now in order to know that which it is necessary to add to this advantage in order to have regard to the expectation that Pierre has to make the hand, it is necessary to determine what is the number which expresses this expectation, & to multiply it by the advantage already found $\frac{3}{17}A$.

It is clear that this expectation is different according to all the different dispositions that the cards of the three coupeurs are able to have. Thus it is necessary to seek what degree of probability there is that each of these possible dispositions will be found, & to multiply each of the numbers which express them by the degree of probability that there is that in such & such disposition Pierre will make the hand.

Now I find that out of twenty-two thousand one hundred different possible dispositions of the three cards of Pierre, Paul, & Jacques, there are eighteen thousand three hundred four, so that the three cards are single; two thousand four hundred ninety-six, so that the card of Pierre is double, one thousand two hundred forty-eight, so that the card of Jacques is double; & fifty-two so that that of Pierre is triple.

It is necessary further to observe, (1) that when the three cards of Pierre, Paul & Jacques are single, there is odds one against two that Pierre will make the hand.

(2) That there is odds three against two, when the card of Pierre is double; & two against three, when the card of Jacques is double.

From all that it follows that if one supposes, for brevity, $\frac{3}{17}A = b$, the expectation that Pierre has to make the hand will be expressed by this quantity,

$$\frac{18304 \times \frac{1}{3}b + 2496 \times \frac{3}{5}b + 1248 \times \frac{2}{5}b + 52 \times b}{22100} = \frac{2351}{6375}b.$$

Therefore if one names h the advantage of Pierre, when one supposes indeterminately that he will continue to hold the cards as many times as he will continue to hold the hand (that which is the sought Problem) one will have $h = \frac{3}{17}A + \frac{2351}{6375} \times \frac{3}{17}A + \frac{2351}{6375} \times \frac{2351}{6375} \times \frac{3}{17}A + \overline{\frac{2352}{6375}}^3 \times \frac{3}{17}A +$&c. The sum of this infinite series is $\frac{19125}{68408}A$, so that if the game is with the pistoles, the advantage of the one who has the hand will be 2 livres 15 sols, & around 10 deniers.

Second Case

I suppose that there are four coupeurs; the fourth is named Jean.

In order to discover in how many different ways the cards of the three coupeurs, Paul, Jacques & Jean are able to arrive either single, or doubles, or triples; it is necessary to remember that in the preceding case one has found that there is odds sixteen against one that the card of the first coupeur being single, that of the second will be it also; & that the cards of two coupeurs being singles, there is odds twenty-two against three that the following card will be single. (2) That the cards of the first two coupeurs being singles, there is odds six against forty-four that the third will be double. (3) That there is odds one against sixteen that that the card of the second coupeur will be double; & that the card of the second coupeur being double, there are two out of fifty in order to bring forth a triple card, & consequently forty-eight out of fifty in order to bring forth single card.

From all that it follows,

(1) That in order to determine how much there be odds that in this case here the cards of the three coupeurs will be singles; it is necessary to multiply the number $\frac{22}{25}$ which expresses the degree of probability that there is that the cards of Paul & of Jacques being singles, that of Jean will be also, by the number $\frac{16}{17}$ which expresses how much the probability be that that of Jacques will be single; thus there is odds three hundred fifty-two against seventy-three, that the cards of the three coupeurs, Paul, Jacques & Jean will be singles. (2) That in order to have the number which expresses how much is the odds that the card of Jean will be double, it is necessary to multiply $\frac{6}{50}$ by the number $\frac{16}{17}$. (3) That in order to have the number which expresses how much is the probability that that of Jacques will be double, & that of Jean single, it is necessary to multiply $\frac{1}{17}$ by the number $\frac{24}{25}$. (4) That the fraction $\frac{1}{17} \times \frac{1}{25}$ expresses how much the odds would be that the card of Jean would be triple.

Now it is necessary to determine what is the lot of Pierre in each of these four different dispositions of the cards of the three Players.

One will find, (1) that the cards of Paul, Jacques & Jean being singles, Pierre out of forty-nine cards which remain, has forty of them to draw which are able to give to him single card, & nine which are able to give to him double card. Now the lot of Pierre when he has single card, the cards of the three other coupeurs being singles also, is $3A$; & his lot, when he has double card, any two from among the coupeurs having single card, is $4A + \frac{2}{5}A$. One will have therefore the lot of Pierre, when the cards of the three other coupeurs are singles, $= \frac{40}{49} \times 3A + \frac{9}{49} \times \overline{4A + \frac{2}{5}A} = \frac{798}{245}A = 3A + \frac{63}{245}A$.

(2) That the card of Jean being double, Pierre has out of forty-nine cards which remain forty-four cards to draw which are able give to him single card, three cards which are able to give to him double card, & finally two cards which are able to give to him triple card. Now the lot of Pierre, when his card is single, is $2A + \frac{3}{5}A$; & his lot, when his card is double, is $4A$. Finally his lot, when his card is triple, is $4A + \frac{3}{2}A$. Because, Pierre having triple card against one other single card, would have to have odds three against one in order to wager equally, & consequently he has three against one on the sum which is laid

on the card which remains: Therefore if the card of Jean is double, the lot of Pierre will be $= \frac{44}{49} \times \overline{2A + \frac{3}{5}A} + \frac{3}{49} \times 4A + \frac{2}{49} \times \overline{4A + \frac{3}{2}A} = \frac{687}{245}A = 2A + \frac{197}{245}A.$

One will find that the lot of Pierre will be the same, that is $2A + \frac{197}{245}A$, when the card of Jacques will be double.

(3) One will observe that the card of Jean being triple, Pierre out of forty-nine cards has forty-eight of them, which give to him single card against triple card, & one alone which gives to him quadruple card.

Now the lot of Pierre when the card is single is $2A$, because he has a coup in order to have $8A$, & three coups in order to have zero. His lot, when his card is quadruple, is $8A$. Therefore if the card of Jean is triple, the lot of Pierre will be $= \frac{48}{49} \times 2A + \frac{1}{49} \times 8A = \frac{104}{49}A = A + \frac{55}{49}A.$

It is necessary further to remark that Pierre risks $3A$ only in the case where the cards of Paul, Jacques & Jean are singles; that he risks only $2A$ in the case where the card, either of Jacques, or of Jean, is double, & only A in the case where the card of Jean is triple.

All this supposed, the advantage that Pierre has in each hand, will be expressed by this quantity,

$$\frac{\overline{352 \times 63 + 72 \times 197 + 55 \times 5 \times} A}{425 \times 245} = \frac{7327}{17 \times 25 \times 49}A = \frac{7327}{20825}A.$$

The concern now is to discover how much be the probability that Pierre will make the hand. In order to come to the end of it, it is necessary to be taken as one has done in the preceding case; to examine what is the number which expresses each of the following dispositions of the four cards. Namely, (1) that all the cards are singles; (2) that the card of Pierre is single, one of the three others being double; (3) that the card of Pierre is double, any two others being singles; (4) that the card of Pierre is double, one of the others being double; (5) that the card of Jean is triple, that of Pierre being single; (6) that the card of Pierre is triple, any one of the three others being single; (7) that the card of Pierre is quadruple; & next to seek what is the expectation of Pierre to make the hand in each of these seven different dispositions of the four cards.

Now I find that expressing the expectation that Pierre has to make the hand in the seven different dispositions above marked by the unknowns x, y, z, u, t, p, l, according to the order that one just gave to them, & designating by the letter b that which reverts to Pierre of this expectation, & by the letter g the advantage of Pierre when one supposes that he will recommence a second time to keep the cards, in case that he may make the hand in the first turn, one will have

$$\begin{aligned} g = b \quad &+ \quad \frac{14080x + \overline{2112 + 1056} \times y + 3168z + \overline{72 + 144} \times z}{20825} \\ &+ \quad \frac{48t + \overline{48 + 96} \times p + 1 \times l}{20825}. \end{aligned}$$

One will find also $x = \frac{1}{4}b$, $y = \frac{11}{40}b$, $z = \frac{9}{20}b$, $u = \frac{1}{2}b$, $t = \frac{1}{4}b$, $p = \frac{3}{4}b$, $l = b$; & substituting these values, one will have

$$\begin{aligned} g &= b + \frac{14080 \times \frac{1}{4}b + 3168 \times \frac{29}{40}b + 216 \times \frac{1}{2}b + 48 \times \frac{1}{4}b + 144 \times \frac{3}{4}b + b}{20825} \\ &= b + \frac{30229}{104125} = \frac{7327}{20825}A + \frac{30229}{104125} \times \frac{7327}{20825}A. \end{aligned}$$

Therefore if one names h the advantage of Pierre, when one supposes indeterminately that he will continue to keep the cards until he has failed to make the hand, one will have

$$h = \frac{7327}{20825}A + \frac{30229}{104125} \times \frac{7327}{20825}A + \overline{\frac{30229}{104125}}^2 \times \frac{7327}{20825}A + \overline{\frac{30229}{104125}}^3 \times \frac{7325}{20825}A + \&\text{c.}$$

THIRD CASE.

One supposes that there are five coupeurs, I name the fifth Thomas, & the rest as before.

When Pierre goes to draw his card, here is all the different dispositions where the cards of the four other coupeurs are able to be found.

(1) The cards of Paul, Jacques, Jean, Thomas, are able to be found singles, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{40}{49} \times \frac{22}{25} \times \frac{16}{17}$.

(2) The card of Thomas is able to be found single, that of Jean being double, & the number which expresses how much the odds would be that this disposition of cards will be found, is $\frac{44}{49} \times \frac{3}{25} \times \frac{16}{17}$.

(3) The card of Thomas is able to be found single, that of Jacques being double, & the number which expresses how much the odds would be that this disposition of cards will be found, is $\frac{44}{49} \times \frac{24}{25} \times \frac{1}{17}$.

(4) The card of Thomas is able to be found single, that of Jean being triple, & the number which expresses how much the odds would be that this disposition of cards will be found, is $\frac{48}{49} \times \frac{1}{25} \times \frac{1}{17}$.

(5) The card of Thomas is able to be found double, the cards of any two others being singles, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{9}{49} \times \frac{22}{25} \times \frac{16}{17}$.

(6) The card of Thomas is able to be found double, that of Jacques being double, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{3}{49} \times \frac{24}{25} \times \frac{1}{17}$.

(7) The card of Thomas is able to be found double, that of Jean being double, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{3}{49} \times \frac{3}{25} \times \frac{16}{17}$.

(8) The card of Thomas is able to be found triple, the card of Jean being single, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{2}{49} \times \frac{24}{25} \times \frac{1}{17}$.

(9) The card of Thomas is able to be found triple, the card of Paul or of Jacques being single, & the number which expresses how much the odds would be that this disposition will be found, is $\frac{2}{49} \times \frac{3}{25} \times \frac{16}{17}$.

(10) The card of Thomas is able to be found quadruple, & the number which expresses how much the odds would be that this disposition will arrive, is $\frac{1}{49} \times \frac{1}{25} \times \frac{1}{17}$.

One would easily be able to demonstrate all this, but it would be necessary to make a long discourse, which being quite abstract, would give too much difficulty to the Readers. The reflections that I have made in the two preceding cases are able to serve as demonstration for the one here to intelligent persons. I have affected not at all to confound the products of the quantities which are multiplied, & to leave them under this form $\frac{40\times22\times16}{49\times25\times17}$, $\frac{44\times3\times16}{49\times25\times17}$, &c. in order to better make known the order & the formation.

One will be able in this manner to discover all the different possible dispositions of the cards of five coupeurs, by being served with the values which have determined those of four coupeurs, likewise as one was just served for the present case by those that one had already found for three coupeurs, in order to determine all the possible different dispositions of the cards of four coupeurs; & to find anew those of six by means of those that one will have found for five coupeurs, & thus consecutively; & consequently this method is as general as it is possible.

It is proper to note that it is not always necessary to know all the variations which are able to be found in the disposition of the cards of the coupeurs, because according to the nature of the Problem, one is able to confound certain of them, & to neglect to consider them separately, that which in certain encounters diminishes extremely the work of the mind & shorten the solution. This remark is very important for the Problem which will follow, it holds also in the one here, where it is necessary to observe that one is able to comprehend in one same fraction the second, the third & the fifth article, the sixth & the seventh, the fourth, the eighth & the ninth, whence it follows that of all the variations which are able to be found among the dispositions of the cards of four coupeurs, there are only five that it is proper to consider, & consequently if one names x the lot of Pierre when the cards of the four other coupeurs are single.

y his lot when there is a double in them, any two others being singles.

t his lot when there are two doubles.

z his lot when there is one triple & one single.

p his lot when the card of Thomas is quadruple.

One will have the lot of Pierre in each hand

$$= \frac{14080x + 6336y + 192z + 216t + p}{10825}$$

In order to know the value of x, one will note that Pierre by drawing in forty-eight cards has thirty-six of them which are able to give to him a single card, & twelve which are able to give to him double card. Now the lot of Pierre when he has single card is $4A$, & his lot when he has double card is $2A + \frac{3}{5} \times 6A$. Thence it follows that $x = \frac{36}{48} \times 4A + \frac{12}{48} \times \overline{5A + \frac{3}{5}A} = \frac{1056}{240}A = 4A + \frac{2}{5}A$.

In order to determine the value of y, one will note that out of forty-eight cards which remain there are two which are able to give triple card to Pierre, six which are able to give to him double card, & consequently forty which are able to give to him single card. Now the lot of Pierre when his card is triple, is $4A + \frac{3}{4} \times 4A = 7A$, & his lot when his card is double, is $5A + \frac{1}{5}A$, & his lot when his card is simple, is $3A + \frac{3}{5}A$. From all this it follows that $y = \frac{2}{48} \times 7A + \frac{6}{48} \times \overline{5A + \frac{1}{5}A} + \frac{40}{48} \times \overline{3A + \frac{3}{5}A} = 3A + \frac{113}{120}A$.

One will find by the similar reasoning

$$\begin{aligned} z &= \frac{1}{48} \times 10A + \frac{3}{48} \times \overline{4A + \frac{2}{3}A} + \frac{44}{48} \times 3A = 3A + \frac{1}{4}A. \\ t &= \frac{4}{48} \times \overline{4A + \frac{2}{3}} \times 4A + \frac{44}{48} \times \frac{2}{5} \times 8A = 3A + \frac{22}{45}A. \\ p &= 0 \end{aligned}$$

From all that one is able to conclude that the advantage of Pierre in each hand is

$$\frac{14080 \times \frac{2}{5}A + 6336 \times \frac{113}{120}A + 192 \times \frac{4}{5}A + 216A + \frac{22}{45}A - A}{20825} = \frac{12159}{20825}A.$$

In order to determine the expectation that Pierre has to make the hand, one will make some reasonings parallel to those of the two preceding cases, & one will find that this expectation is expressed by the fraction $\frac{106994}{437325}$. This supposed, the sought advantage of Pierre will be $\frac{12159}{20825}A + \frac{106994}{437325} \times \frac{12159}{20825}A + \overline{\frac{106994}{437325}}^2 \times \frac{12159}{20825}A + \overline{\frac{106994}{437325}}^3 \times \frac{12159}{20825}A$+&c.

FOURTH CASE.

One supposes that there are six coupeurs, I name the sixth André, & the rest as before.

If one names x the lot of Pierre when the cards of the five coupeurs are singles.
y his lot when the one of the five is double, the others being singles.
z his lot when there is one triple & two singles.
t his lot when there are two doubles.
q his lot when there is one triple & one double.
p his lot when the last player, who is here André, has a quadruple card.
f his lot when Thomas, who is the penultimate player, has a quadruple card.
One will have the lot of Pierre in each hand

$$= \frac{10560x + 8800y + 440z + 990t + 30q + 4p + f}{20825}$$

One will find also

$$\begin{aligned} x &= 5A + \frac{27}{47}A \\ y &= 5A + \frac{26}{235}A \\ z &= 4A + \frac{96}{235}A \\ t &= 4A + \frac{92}{141}A \\ q &= 3A + \frac{227}{235}A \\ p &= A \\ f &= 0; \end{aligned}$$

that which will give the advantage of Pierre in each hand

$$\begin{aligned} &= \frac{10560 \times \frac{27}{47}A + 8800 \times \overline{A + \frac{26}{235}A} + 440 \times \overline{A + \frac{96}{235}A} + 990 \times \overline{A + \frac{92}{141}A} + 30 \times \overline{A + \frac{227}{235}A} + 5 \times -A}{20825} \\ &= \frac{170607}{195755}A. \end{aligned}$$

One will find by some rather long calculations, but parallel to those of the preceding cases, that the expectation that Pierre has to make the hand, is expressed by the fraction $\frac{1899236042}{10473828825}$, so that one will have the sought advantage $= \frac{170607}{195755}A + \frac{1899236042}{10473828825} \times \frac{17067}{195755}A + \overline{\frac{1899236042}{10473828825}}^2 \times \frac{17067}{195755}A + \overline{\frac{1899236042}{1043828825}}^3 \times \frac{170607}{195755}A$+&c.

FIFTH CASE.

One supposes that there are seven coupeurs.

Let x be the advantage of Pierre when the six cards are singles.
y when there is one double & four singles.

z when there are two doubles & two singles.
u when there are three doubles.
t when there is one triple & three singles.
r when there is one triple, one double & one single.
p when there are two triples.
q when there is one quadruple & two singles.
m when there is one quadruple & one double.
One will have the advantage of Pierre in each hand

$$= \frac{67584x + 95040y + 23760z + 594u + 7040t + 1584r + 12p + 132q + 9m}{195755}$$

$x = \frac{18}{23}A$, $y = \frac{151}{115}A$, $z = \frac{638}{345}A$, $u = \frac{55}{23}A$, $t = \frac{184}{115}A$, $r = \frac{99}{46}A$, $p = \frac{58}{23}A$, $q = -A$, $m = 0$.

Therefore if one substitutes these values of x, y, z, &c. one will have the advantage of Pierre in each hand

$$= \frac{\begin{array}{c} 67584 \times \frac{18}{23}A + 95040 \times \frac{151}{115}A + 23760 \times \frac{618}{345}A + 594 \times \frac{55}{23}A \\ +7040 \times \frac{184}{115}A + 1584 \times \frac{99}{46}A + 12 \times \frac{58}{23}A + 132 \times -A + 9 \times 0 \end{array}}{195755}$$

$$= \frac{5465122}{4502365}A = A + \frac{962757}{4502365}A.$$

One will find that the expectation that Pierre has to make the hand, is expressed by the fraction $\frac{917160030257719}{6102195875135235}$, & consequently the sought advantage will be

$$\frac{5465122}{4502365}A + \frac{917160030257719}{6102195875135235} \times \frac{5465122}{4502365}A + \overline{\frac{917160030257719}{6102195875135235}}^2 \times \frac{5465122}{4502365}A + \overline{\frac{917160030257719}{6102195875135235}}^3 \times \frac{5465122}{4502365}A + \&c.$$

One will be able thus to find the advantage of Pierre, by supposing that there are a greater number of coupeurs, the method for it would be the same, but the calculations for it would be so long, & the reasoning which the calculations suppose so troubled, that I believe I must dispense going further; it is rare that there are more than seven coupeurs, & the utility that one would be able to draw from a table calculated for a greater number of coupeurs would not be in our opinion considerable enough in order to compensate for the difficulty that it would give.

I am going presently to give the solution of another part of the Problem that I have proposed to myself on Lansquenet, namely to determine the diverse advantages of the coupeurs who are in the different places to the right & to the left of Pierre. The method that I will employ will have much relation to the preceding, thus in order to make it understood I will content myself to make the test & application of it on a particular case such as is the one which follows.

PROPOSITION VI.
PROBLEM II.

To determine what is the ratio of the different disadvantages of three coupeurs, Paul, Jacques & Jean, in supposing as in the second case of the preceding Problem, that fourth coupeur Pierre holds the hand, that Paul is the first to his right, that Jacques follows Paul, & that Jean is to the left of Pierre.

One has found in the solution of the second case of the preceding Problem, page 25, that the advantage of Pierre in each hand was expressed by the fraction $\frac{7327}{20825}$, so that the game being with pistoles, he must estimate his advantage three livres ten sols & some deniers. Now it is clear that this advantage of Pierre falls into loss on the other coupeurs, but unequally for each, in a way, for example, that Paul bears more of it than Jacques, & Jacques more than Jean.

The difficulty of the Problem consists in discovering according to what proportion this loss or this common disadvantage is distributed on each of the three coupeurs.

In order to find this ratio I seek separately the disadvantage of each of the three players, & for this I examine all the possible dispositions of the four right cards which vary the lot of each of the players, & I observe in each what is his disadvantage, having regard to that which the distribution of money gives to him or makes hope of gain or loss. I multiply each of the numbers which express the different dispositions of cards which vary the condition of the player by the advantage or the disadvantage that they give to him; I add all these products, & I divide their sum by 20825, which is the product from these three numbers 17, 25, 49; the exponent of this division expresses the disadvantage of this player.

To find the disadvantage of Paul.

(1) When the card of the four coupeurs are singles, there is neither advantage nor disadvantage for Jacques.
(2) When the card of Pierre is double, those of Jacques & of Jean being singles, the disadvantage of Paul is expressed by $-3A$.
(3) When the card of Pierre is double, that of Paul being single, the advantage of Paul is $\frac{4}{5}A$.
(4) When the card of Jacques is double, the disadvantage of Paul is expressed by $-A$.
(5) When the card of Jean is double, that of Paul being in loss, the disadvantage of Paul is expressed by $-2A$.
(6) When the card of Jean is double, those of Pierre & of Paul being singles, the advantage of Paul is $\frac{4}{5}A$.
(7) When the card of Jean is double of that of Jacques, & the card of Pierre is double of that of Paul, the disadvantage of Paul is expressed by $-A$.
(8) When the card of Pierre is triple, that of Paul being single, the advantage of Paul is $\frac{3}{2}A$.
(9) When the card of Jean is triple, the disadvantage of Paul is expressed by $-A$.

The numbers which express how much the probability is that each of these particular dispositions will be found, make by commencing with the second, & by continuing with order. 352×3, 352×6, 24×49, 24×49, 24×44, 24×3, 24×2, 49, & consequently the disadvantage of Paul will be expressed by this quantity

$$\frac{\begin{array}{l}352 \times 3 \times 5 \times -3A + 352 \times 6 \times 4A + 24 \times 49 \times 5 \times -3A + 24 \times 44 \times 4A \\ +72 \times 5 \times -A + 72 \times 5 \times A + 49 \times 5 \times -A\end{array}}{20825 \times 5}$$

which being reduced becomes $-\frac{21053}{104125}A$; & this fraction expresses the disadvantage of Paul.

To find the disadvantage of Jacques.

(1) When the cards of the four coupeurs are singles, there is neither advantage nor disadvantage for Paul.
(2) When the card of Pierre being double, those of Paul & of Jean are singles, the disadvantage of Jacques is expressed by $-3A$.
(3) When the card of Pierre being double, that of Jacques is single, the advantage of Jacques is $\frac{4}{5}A$.
(4) When the card of Jean being double, those of Jacques & Pierre are singles, the advantage of Jacques is $\frac{4}{5}A$.
(5) When the card of Jean being double of that of Paul, the card of Pierre is double of that of Jacques, the disadvantage of Jacques is expressd by $-A$.
(6) When the card of Jean being double, those of Paul & of Pierre are singles, the disadvantage of Jacques is expressed by $-2A$.
(7) When the card of Jean being double, that of Pierre is double of the card of Paul, the disadvantage of Jacques is expressed by $-2A$.
(8) When the card of Jacques being single, that of Pierre is triple, the advantage of Jacques is $\frac{3}{2}A$.
(9) When the card of Paul being single, that of Pierre is triple, the disadvantage of Jacques is expressed by $-2A$.
(10) When the card of Jacques being double, those of Jean & of Pierre are singles, his advantage is $\frac{3}{5}A$.
(11) When the card of Jacques being double, that of Pierre is double, the advantage of Jacques is A.
(12) When the card of Jean being single, that of Pierre is triple, the disadvantage of Jacques is expressed by $-3A$.
(13) When the card of Pierre is quadruple, the disadvantage of Jacques is expressed by $-2A$.
(14) When the card of Pierre being single, that of Jean is triple, the disadvantage of Jacques is expressed by $-2A$.

The numbers which express how much probability there be that each of these particular dispositions will be found, are by commencing with the second, & by continuing with order, 3×352, 6×352, 24×44, 24×3, 24×44, 24×3, 24×2, 24×2, 24×44, 24×3, 24×2, 1, 48, & consequently the disadvantage of Jacques will be expressed by this quantity

$$\frac{\begin{array}{l}3 \times 352 \times -3A + 6 \times 352 \times \frac{4}{5}A + 24 \times 44 \times \frac{4}{5}A + 24 \times 3 \times -A \\ +24 \times 44 \times -2A + 24 \times 3 \times -2A + 24 \times 2 \times \frac{3}{2}A + 24 \times 2 \times -2A \\ +24 \times 44 \times \frac{3}{5}A + 24 \times 3A + 24 \times 2 \times -3A + 1 \times -2A + 48 \times -2A\end{array}}{20825}$$

That which being reduced gives this fraction $-\frac{12610}{104125}A$, which expresses the disadvantage of Jacques.

To find the disadvantage of Jean.

(1) When the card of the four coupeurs are singles, he has neither advantage nor disadvantage for Jean.
(2) When the card of Jean being double, that of Pierre is single, the advantage of Jean is $\frac{3}{5}A$.
(3) When the card of Pierre & that of Jean are singles, that of Jacques being double, the disadvantage of Jean is expressed by $-\frac{1}{5}A$.
(4) When the card of Pierre being single, that of Jean is triple, the disvantage of Jean is $2A$.
(5) When the card of Pierre being double, those of Paul & of Jacques are singles, the disadvantage of Jean is expressed by $-3A$.
(6) When the card of Pierre being double, that of Jean is single, the advantage of Jacques is $\frac{4}{5}A$.
(7) When the card of Pierre being double, that of Jacques is double, the disadvantage of Jean is expressed by $-2A$.
(8) When the card of Pierre being double, that of Jean is double, the advantage of Jean is A.
(9) When the card of Pierre being triple, that of Jean is single, the advantage of Jacques is $\frac{1}{2}A$.
(10) When the card of Pierre being triple, that of Paul or of Jacques are singles, the disadvantage of Jean is expressed by $-3A$.
(11) When the card of Pierre is quadruple, the disadvantage of Jean is expressed by $-4A$.

The numbers which express the probability that there be that each of these particular dispositions will be found, are by beginning with the second, & by continuing with order, $44 \times 3 \times 16$, $44 \times 24 \times 1$, 48, $3 \times 22 \times 16$, $6 \times 22 \times 16$, 3×24, $3 \times 3 \times 16$, 2×24, $2 \times 3 \times 16$, 1, An consequently the disadvantage of Jean will be

$$\frac{\begin{array}{l}44 \times 3 \times 16 \times \frac{3}{5}A + 44 \times 24 \times -\frac{1}{5}A + 48 \times 2A + 3 \times 22 \times 16 \times -3A \\ +6 \times 22 \times 16 \times \frac{4}{5}A + 3 \times 24 \times -2A + 3 \times 3 \times 16 \times A \\ +2 \times 24 \times \frac{1}{2}A + 2 \times 3 \times 16 \times -3A + 1 \times -4A\end{array}}{20825}$$

that which being reduced becomes $-\frac{2972}{104125}A$, & this fraction expresses the disadvantage of Jean.

Now if one adds into one sum the disadvantages found of the three players Paul, Jacques & Jean, $\frac{-21053-12610-2972 \times A}{20825 \times 5}$, one will find that their sum $= -\frac{7327}{20825}A$.

Now one has seen in the second case of the preceding Problem, that the advantage of Pierre in each hand was $\frac{7327}{20825}A$, & consequently these two terms being compared, are destroyed.

One has therefore the just proportion of the disadvantage of the three players, & the total of their disadvantage, thus as one has ought to find it.

One has paid here attention only to the disadvantage that each of the players Paul, Jacques & Jean has in each hand. Now if one wishes to have regard to that which occurs to them as disadvantage when one supposes that Pierre will recommence to hold the cards as many times as he will make the hand, one will find the disadvantage of Paul,

$= -1 \times \frac{21053}{104125}A + \frac{30229}{104125} \times \frac{21053}{104125}A + \overline{\frac{30229}{104125}}^2 \times \frac{21053}{104125}A + \overline{\frac{30229}{104125}}^3 \times \frac{21053}{104125}A +$ &c.
& the one of Jacques
$= -1 \times \frac{12610}{104125}A + \frac{30229}{104125} \times \frac{12610}{104125}A + \overline{\frac{30229}{104125}}^2 \times \frac{12610}{104125}A + \overline{\frac{30229}{104125}}^3 \times \frac{12610}{104125}A +$ &c.
& the one of Jean $= -1 \times \frac{2972}{104125}A + \frac{30229}{104125} \times \frac{2972}{104125}A + \overline{\frac{30229}{104125}}^2 \times \frac{2972}{104125}A + \overline{\frac{30229}{104125}}^3 \times \frac{2972}{104125}A +$ &c.
The sum of these three infinite series will be equal to that which expresses the advantage of Pierre, & being compared to it, they are destroyed having contrary signs.

COROLLARY I.

If one wishes to know the exact values of the infinite series which express the advantage of the one who holds the hand, by supposing that A which expresses the game is a pistole or ten livres; one will have then in this table.

For three coupeurs his advantage will be	2 l. 15 s. 10 d. $\frac{490}{503}$.
For four coupeurs	4 l. 19 s. 1 d. $\frac{2569}{3079}$.
For five coupeurs	7 l. 14 s. 7 d. $\frac{4955}{330331}$.
For six coupeurs	10 l. 12 s. 10 d. $\frac{328372137818918}{335703882047233}$.
For seven coupeurs	14 l. 16 s. 5 d. $\frac{1276210397023}{7756210003115777}$.

It follows thence that the advantage of the one who holds the hand does not increase in the same ratio as the number of players, since his advantage which is around 2 liv. 16 sols when there are three coupeurs, is much greater than 5 liv. 12 sols when there are six coupeurs.

COROLLARY II.

If one supposes that the game is in pistoles, & that there are four coupeurs Pierre, Paul, Jacques & Jean, thus as in the second Problem, or in the second case of the first, the

disadvantage of Paul will be	2 l. 16 s. 11 d. $\frac{2343}{3079}$;
the disadvantage of Jacques will be	1 l. 14 s. 1 d. $\frac{1689}{3079}$;
the disadvantage of Jean will be	8 s. 0 d. $\frac{1616}{3079}$;

It is necessary to note, (1) that the sum of the the three terms which express the diverse disadvantages of the players is equal to the one here 4 liv. 19 s. 1 d. $\frac{2569}{3079}$, which expresses the advantage of Pierre; (2) that the ratio of the disadvantages of Paul, Jacques & Jean is very nearly as 7, 4, 1.

COROLLARY III.

The probability that there is that Pierre will make the hand, diminishes in measure as there are a greater number of coupeurs; And the order of this diminution from three coupeurs to seven inclusively, is very nearly as these fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$.

COROLLARY IV.

There is found often some coupeurs who for lack of knowing their interests, or by an imagination that they have of having the unlucky hand, or finally in order to not lose more money than they have planned to risk, pass their hand without quitting the game. Each coupeur will know by the second Problem, how much the one who renounces the hand gives advantage to him.

Corollary V.

It is likewise when a coupeur quits the game, each of the other coupeurs will be able to discover by the same Problem, how much that is advantageous or prejudicial to him.

REMARK I.

It is a common prejudice among the Players, that the card of the réjouissance is favorable to those who set there. In order to be disabused of this opinion, it is necessary to take care that if the card of the réjouissance has some advantage in certain dispositions of the cards of the coupeurs, it has some disadvantage in others of them, & that that is compensated always exactly.

Let us suppose, for example, that there are three coupeurs as in the first case of the first Problem, & that the money of the réjouissance is named $2b$, it is quite true that the advantage of the réjouissance will be $\frac{6}{245}b$ when the cards of the three coupeurs will be singles, & $\frac{1}{49}b$ when the card of Jacques will be double: but in recompense his disadvantage will be $\frac{44}{245}b$ when the card of Pierre will be double, & $\frac{24}{49}b$ when it will be triple.

Multiplying therefore these numbers by those which express the different probabilities that there are that such or such of these dispositions will be encountered, one will have $\frac{6\times352}{245\times425}b+\frac{24}{49\times425}b-\frac{44\times48}{245\times425}b-\frac{24}{49\times425}b=0$; that which shows that there is in this case neither advantage, nor disadvantage for the card of the réjouissance.

One will be able to discover the same thing with respect to every other number of coupeurs.

REMARK II.

There is a Game known enough that one names the Duppe, it is a kind of reversed Lansquenet. The difference of this Game to the one of Lansquenet consists in that which follows; (1) the one who holds the Duppe, is given the first card; (2) the one who has cut the cards is obliged to take the second; (3) the other Players are able to take or refuse the card that is presented to them; (4) the one who takes a double card is obliged to make the part of it; (5) the one who holds the Duppe quits these cards not at all, & always conserves the hand. The resemblance that there is of this Game to the one of Lansquenet, has made the Players imagine that there is disadvantage for the one who holds the hand, & so much more, as in this Game the hand changes not at all, instead as in Lansquenet each holds it in his turn. On this foundation they have given to him the name of the Duppe: but there arrives to him nothing, because it is easy to discover that the equality is perfect in this Game & for the Players among them, & for the one who holds the hand in regard to the Players. It suffices for me to make this remark, a little attention on it will convince those who would wish to take the pain to examine it.

DIVERSE PROBLEMS
ON THE GAME
OF TREIZE

EXPLICATION OF THE GAME

The players draw first for who will have the hand. Let us suppose that it is Pierre, & that the number of the players is such as one will wish. Pierre having an entire deck composed of fifty-two cards shuffled at discretion, draws them one after the other. Naming & pronouncing one when he draws the first card, two when he draws the second, three when he draws the third, & thus in sequence up to the thirteenth which is a King. Now if in all this sequence of cards he has drawn none of them according to the rank as he has named them, he pays that which each of the players has set into the game, & gives the hand to the one who follows him at the right.

But if it happens to him in the sequence of thirteen cards, to draw the card which he names, for example, to draw one ace at the time which he names one, or a two at the time which he names two, or a three at the time which he names three, &c. he takes all that which is in the game, & restarts as before, naming one, next two, &c.

It is able to happen that Pierre having won many times, & restarting with one, has not enough cards in his hand in order to go up to thirteen, now he must, when the deck falls short to him, to shuffle the cards, to give to cut, & next to draw from the entire deck the number of cards which is necessary to him in order to continue the game, by commencing with the one where he is stopped in the preceding hand. For example, if drawing the last card from them he has named seven, he must in drawing the first card from the entire deck, after one has cut, to name eight, & next nine, &c. up to thirteen, unless he rather not win, in which case he would restart, naming first one, next two, & the rest as we just explained it. Whence it seems that Pierre is able to make many hands in sequence, & likewise he is able to continue the game indefinitely.

The advantage is quite considerable in this Game in favor of the one who holds the hand, & those who play it often are able to perceive it by practice; but it is extremely difficult to determine this advantage: Analysis would be able to lead there, but this route would be extremely long, & I find that it would be necessary to resolve more than a thousand equalities in order to determine all the possible cases of the Game. One would be able rather to hope to the solution by considering all the possible arrangements of the fifty-two cards, & discovering as one has done for Pharaon, & as one will do in the Problem following on the game of Bassette, some uniform law, which leads from the simple cases to some more composed cases, & furnishes thus a general solution. I will not give the solution of this Problem at all, but in its place here are two which have of much relation to it, & of which the solution will be able to facilitate that of the game of Treize to those of among my Readers who would wish to take the pain to do the research on it.

PROBLEM
PROPOSITION VII.

Pierre has a certain number of different cards which are not repeated at all, & which are shuffled at discretion: he bets against Paul that if he draws then in sequence, & if he names them according to the order of the cards, beginning of them either with the highest, or with the lowest, there will happen to him at least one time to draw the one that he will name. For example, Pierre having in hand four cards, namely an ace, a deuce, a three & a four shuffled at discretion, wagers that drawing them in sequence, & naming one when he will draw the first, two when he will draw the second, three when he will draw the third, there will happen to him either to draw one ace when he will name one, or to draw a deuce when he will name two, or to draw a three when he will name three, or to draw a four when he will name four. Let be imagined the same thing for each other number of cards. One asks what is the lot or the expectation of Pierre for whatever number of cards that this may be from two up to thirteen.

SOLUTION

Let the cards with which Pierre makes the part, be represented by the letters a, b, c, d, &c. If one names m the number of cards which he holds, & n the number which expresses all the possible arrangements of these cards, the fraction $\frac{n}{m}$ will express how many different times each letter will occupy each of the positions. Now it is necessary to note that these letters are not encountered always in their place advantageously for the Banker; for example, a, b, c only gives a winning coup to the one who has the hand, although each of these three letters be in its place there; And similarly b, a, c, d gives only one winning coup to Pierre, although each of the letters c & d be in its place there. The difficulty of this Problem consists therefore in untangling how many times each letter is in its place advantageously for Pierre, & how many times it is useless to him.

It would be too long to put into detail the reflections which have led me to the solution of this Problem. I am going to give first succinctly the solution of the first cases which are the simplest, next I will give a general formula, & I will form a table which will express the lot of Pierre in all the different cases, from two cards to thirteen inclusively.

FIRST CASE.

Pierre holds an ace & a deuce, & bets against Paul, that having shuffled these two cards, & naming one when he will draw the first, & two when he will name the second, there will happen to him either to draw an ace for the first card, or to draw a deuce for the second card. The money of the game is expressed by A.

Two cards are able to be arranged only in two different ways: the one makes Pierre win, the other makes him lose: therefore his lot will be $\frac{A+0}{2} = \frac{1}{2}A$.

SECOND CASE.

Pierre holds three cards.

Let there be three cards represented by the letters a, b, c: one will observe that of the six different arrangements that these three letters are able to admit, there are two of them where a is in the first place; that there is one of them where b is in the second place; a being not at all in the first, & one where c is in the third place, a not at all in the first, & b

not at all in the second; whence it follows that one will have $S = \frac{2}{3}A$; & consequently that the lot of Pierre is to that of Paul, as two is to one.

THIRD CASE.

Pierre holds four cards.

Let the four cards be represented by the letters a, b, c, d: one will observe that of the twenty-four different arrangements that these four letters are able to admit, there are six of them where a occupies the first place; that there are four of them where b is in the second, a not being in the first; three where c is in the third, a not being in the first, & b not being in the second; finally two where d is in the fourth, a not being in the first, b not being in the second, & c not being in the third; whence it follows that one will have the lot of Pierre

$$= S = \frac{6+4+3+2}{24}A = \frac{15}{24}A = \frac{5}{8}A;$$

& consequently that the lot of Pierre is to the lot of Paul as five to three.

FOURTH CASE.

Pierre holds five cards.

Let the five cards be represented by the letters a, b, c, d, f: one will observe that of the 120 different arrangements that five letters are able to admit, there are twenty-four where a occupies the first place, eighteen where b occupies the second, a not occupying the first; fourteen where c is in the third place, a not being in the first place, nor b in the second; eleven where d is in the fourth place, a not being in the first, nor b in the second, nor c in the third; finally nine arrangements where f is in the fifth place, a not being in the first, nor b in the second, nor c in the third, nor d in the fourth; whence it follows that one will have the lot of Pierre

$$= S = \frac{24+18+14+11+9}{120}A = \frac{76}{120}A = \frac{19}{30}A;$$

& consequently that the lot of Pierre is to the lot of Paul as nineteen is to eleven.

GENERALLY

If one names S the lot that one seeks, the number of cards that Pierre holds being expressed by p; g the lot of Pierre, the number of cards being $p-1$; d his lot, the number of cards that he holds being $p-2$, one will have

$$S = \frac{g \times \overline{p-1} + d}{p}.$$

This formula which is very simple & very general, will give all the cases, thus as one sees them resolved in the Table adjoined here.

TABLE

If $p = 1$,	one will have	$S = A.$
If $p = 2$,	one will have	$S = \frac{1}{2}A.$
If $p = 3$,	one will have	$S = \frac{2}{3}A = \frac{1}{2}A + \frac{1}{6}A.$
If $p = 4$,	one will have	$S = \frac{5}{8}A = \frac{1}{2}A + \frac{1}{8}A.$
If $p = 5$,	one will have	$S = \frac{19}{30}A = \frac{1}{2}A + \frac{2}{15}A.$
If $p = 6$,	one will have	$S = \frac{91}{144}A = \frac{1}{2}A + \frac{19}{144}A.$
If $p = 7$,	one will have	$S = \frac{531}{840}A = \frac{1}{2}A + \frac{111}{840}A.$
If $p = 8$,	one will have	$S = \frac{3641}{5760}A = \frac{1}{2}A + \frac{761}{5760}A.$
If $p = 9$,	one will have	$S = \frac{28673}{45360}A = \frac{1}{2}A + \frac{5993}{45360}A.$
If $p = 10$,	one will have	$S = \frac{28319}{44800}A = \frac{1}{2}A + \frac{5919}{44800}A.$
If $p = 11$,	one will have	$S = \frac{2523223}{3991680}A = \frac{1}{2}A + \frac{527383}{3991580}A.$
If $p = 12$,	one will have	$S = \frac{302786759}{479001600}A = \frac{1}{2}A + \frac{63285959}{479001600}A.$
If $p = 13$,	one will have	$S = \frac{109339663}{172972800}A = \frac{1}{2}A + \frac{22853263}{172972800}A.$

It is easy to see that this formula would give the same advantage to Pierre, if one would suppose that he had there a greater number of cards of different kind.

REMARK I.

The preceding solution furnishes a singular usage of the figurate numbers, of which I will speak in the following, because I find on examining the formula, that the lot of Pierre is expressed by an infinite sequence of terms which have alternately $+\&-$, & such that the numerator is the sequence of numbers which compose in the Table, page 54, the perpendicular column which corresponds to p, beginning with p, & the denominator the sequence of products $p \times p-1 \times p-2 \times p-3 \times p-4 \times p-5$, &c. in such a way that these products which are found in the numerator & in the denominator destroying themselves, there remains for expression of the lot of Pierre this very simple series

$$\frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \text{\&c.}$$

(These points are here & will be in the following the mark of the products) If one forms a logarithm of which the subtangent be unity, & if one takes two ordinates, of which the one is unity, & the other is extended to this first by a quantity equal to the subtangent, the excess of the constant ordinate over the last will be equal to this series.

In order to demonstrate it let the general formula of the subtangent be

$$s = \pm\frac{ydx}{dy},$$

the subtangent being named s, the abscissa x, the ordinate y. One will suppose y equal to a series of powers of x affected with indeterminate coefficients, for example,

$$= 1 + ax + bxx + cx^3 + dx^4 + \text{\&c.}$$

& taking on all sides the difference, dividing next by dx, & multiplying by s, one will find

$$\begin{aligned} \pm\frac{sdy}{dx} &= y = 1 + ax + bxx + cx^3 + dx^4 + \text{\&c.} \\ &= \pm\, as \pm 2bsx \pm 3csxx \pm 4dsx^3 + \text{\&c.} \end{aligned}$$

If one compares the homologous terms of these two series, & if one draws from this comparison the value of the coefficients a, b, c, d, one will have

$$y = 1 \pm \frac{x}{s} \pm \frac{1xx}{1.2ss} \pm \frac{1x^3}{1.2.3s^3} \pm \frac{1x^4}{1.2.3.4s^4} \pm \&\text{c.}$$

that which shows that if one determines, y to be the ordinate of a logarithm of which the constant subtangent be $= 1$, one will have the ordinate which corresponds to x taken on the side that the ordinate decreases,

$$= 1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \&\text{c.}$$

one is able to see this demonstration in the Actes of Leipzig for the year 1693, p. 179, where the celebrated Mr. Leibnitz resolves this Problem: *A logarithm being given, to find the number which corresponds to it*. Now it is clear that if in this series one supposes $x = 1$, that is to say equal to the subtangent or to the constant ordinate, & if one subtracts this series from unity, it will become the series of the present Problem.

One is able again to demonstrate it more simply in this manner. Let be imagined a logarithm of which the subtangent is unity; one will take on this curve a constant ordinate $= 1$, & another smaller ordinate $= 1 - y$, one will name x the abscissa contained between the two ordinates, one will have $dx = \frac{dy}{1-y}$, and

$$x = y + \frac{1}{2}yy + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \&\text{c.}$$

& by the method for the reversion of series,

$$y = x - \frac{xx}{1.2} + \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} - \&\text{c.}$$

that which, in supposing $x = 1$, becomes

$$= 1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \&\text{c.} \qquad \text{Q. E. D.}$$

REMARK II.

Although the series of the Problem is composed of an infinite number of terms, it is not at all equal to unity; one will find likewise that it will never be so great as $\frac{1}{2} + \frac{1}{7}$, the sole case excepted where there are three cards, so that the limits of the series are between $\frac{10}{16}$ & $\frac{19}{30}$. One will be able further to find them more correct if one adds into one same sum a greater number of terms of the series, that which is done without pain, by employing the Table of logarithms; & thus the excess or the deficit of this sum with respect to the value of the series will diminish to infinity.

One is able to observe that the series

B $$\frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \&\text{c.}$$

is equal to each of the three C, D, F[1] which follow, which under some very different forms do not omit having the same value; in such a way that all that which agrees to the series B

[1] Series B, D, F all sum to $1 - \frac{1}{e}$. Montmort errors with series C for it sums to $1 + \frac{1}{e}$.

agrees to them also.

$$\text{C} \quad \frac{1}{1.2} + \frac{4}{1.2.3} + \frac{9}{1.2.3.4} + \frac{16}{1.2.3.4.5} + \frac{25}{1.2.3.4.5.6} + \frac{36}{1.2.3.4.5.6.7} + \&c.$$
$$- 2 \times \frac{1}{2} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7.8}$$
$$+ \frac{1}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$$\text{D} \quad \frac{1}{2} + \frac{3}{1.2.3.4} + \frac{5}{1.2.3.4.5.6} + \frac{7}{1.2.3.4.5.6.7.8} + \frac{9}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$$\text{F} \quad \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} + \&c.$$
$$-\frac{1}{3.4} - \frac{1}{3.4.5.6} - \frac{1}{3.4.5.6.7.8} - \frac{1}{3.4.5.6.7.8.9.10} - \frac{1}{3.4.5.6.7.8.9.10.11.12} - \&c.$$

One could make many curious remarks on the relation of these series; but that would deviate us from our subject, & would lead us too far.

One has supposed in this Problem, that Pierre having won or lost, the game would end. But if one supposed, thus as it is practiced in the game of Treize, that Paul having lost once, remits again into the game, & that having lost a second time, he remits again the same sum in the game, & thus consecutively, until this that Pierre had lost by drawing all his cards without naming a single one of them at its rank: This would be a new Problem which would have more difficulty that the one of which we just gave a general solution, & which would be more in relation to the game of Treize. Here is the simplest case of it.

PROBLEM
PROPOSITION VIII.

Pierre plays against Paul under the same conditions as in the game of Treize explicated above, with this sole difference, that instead as in the game of Treize Pierre plays with a deck of fifty-two cards, composed of four aces, four deuces, four threes, &c. here Pierre plays only with thirteen cards, namely an ace, a deuce, a three, a four, &c. to the King exclusively.

First Case.

Pierre holds an ace & a deuce.

I suppose that Pierre & Paul each set into the game a certain sum that I name A. I express the two cards by two letters, namely the ace by the letter a, & the deuce by the letter b. Thus supposed, I examine that which the two different arrangements ab, ba give to Pierre. Now I see that the arrangement ba makes Pierre lose, & that the other arrangement ab puts him in a situation that I see in truth is very favorable to him, but which is unknown to me; since Pierre, in order to finish is obliged to shuffle the cards, & to restart. Now in restarting it is equally able to happen to him, either to lose that which he would have already won, if the cards are found arranged such as the arrangement ab represent it; or to win anew $2A$, with the right to restart, if the cards are disposed such as the arrangement ba represents it; because in this disposition he will win with b, having to name a deuce; & next by a, having to name an ace; & there will be still the right to continue the game, after having shuffled the cards anew.

Therefore naming S the sought advantage of Pierre, x his lot when he has brought forth for the first card an ace, one will have

$$S = \frac{1}{2} \times x + \frac{1}{2} \times 0, x = \frac{1}{2} \times \overline{4A + S} + \frac{1}{2} \times A,$$

whence one draws $B = \frac{5}{3}A$; that which shows that under the supposition of this first case the advantage of Pierre would be expressed by $\frac{2}{3}A$.

SECOND CASE.

Pierre holds an ace, a deuce & a three.

Expressing as above the ace, the deuce & the three by the letters a, b, c, & the sought lot of Pierre by S, I arrange in order to make me understand in fewer words & more easily, the six different arrangements that these three letters are able to receive on three columns, & I set beside that which each of these arrangements give to Pierre of gain, of loss, or of expectation. Now there are of them which leave the fortune of Pierre undecided. Pierre being obliged to finish, to shuffle the cards, & to continue to draw either one card if there remains to him three to name, or two cards if there remains to him two & three to name. In this case I express the lot of Pierre by a variable, & in order to determine it I set anew these same arrangements, & I express beside that which each gives to Pierre. It would be too long to render reason of all in detail, it will suffice to the Reader to regard the Table & to consider with attention if the value of each arrangement is well determined.

$$\begin{array}{rrr} x.abc & 2A+S.bac & 0.cab \\ 3A+S.acb & 0.bca & 3A+S.cba \end{array}$$

I name x the lot of Pierre when his cards are found disposed thus as the arrangement abc represents it. In order to determine it I make this new Table,

$$\begin{array}{rrr} A.abc & A.bac & y.cab \\ A.acb & A.bca & A+x.cba \end{array}$$

I name y the lot of Pierre when his cards being found first disposed thus as the arrangement abc represents it, they are in the second time, thus as the arrangement cab represents it.

In order to determine this unknown I make this third Table.

$$\begin{array}{rrr} 3A.abc & 2A+y.bac & 3A.cab \\ A+y.acb & 3A+x.bca & 3A.cba \end{array}$$

From all this one draws

$$\begin{aligned} S &= \frac{1 \times \overline{2A+S} + 2 \times \overline{3A+S} + 2 \times 0 + 1 \times x}{6} = \frac{8A+3S+x}{6} \\ x &= \frac{5A+x+y}{6} \\ y &= \frac{3 \times 3A + 1 \times 3A + 2y + y + 1 \times 3A + x}{6} = \frac{15A+2y+x}{6} = \frac{15A+x}{4}; \end{aligned}$$

And substituting this value of y in the equality $x = \frac{5A+x+y}{6}$, one will have $x = \frac{5A+x\frac{15A+x}{4}}{6}$; that which gives $x = \frac{35}{19}A$. And substituting this value of x into the equality $S = \frac{8A+3+x}{6}$, one will have $S = \frac{8A+3S+\frac{35}{19}A}{6} = \frac{152A+57S+35A}{114}$.

And finally $S = \frac{187}{57}A = 3A + \frac{16}{17}A$. Whence it appears that the advantage of Pierre would be $2A + \frac{16}{57}A$. So that if A expresses one pistole, that is, if Paul is obliged to put into the game one pistole against Pierre, until this that Pierre losing, the game is ended; the advantage that it gives to Pierre, & the loss that he suffers by taking this part, is the same as if he gave to Pierre in pure gift twenty-two livres thirteen sols & some deniers.

This method is general; but as the length of the calculation renders it impractical when there are a greater number of cards, one must be content only in the meantime by awaiting until one has found a better.

PROBLEMS
ON THE GAME
OF BASSETE

EXPLICATION OF THE RULES

In this game, as in that of Pharaon, the Banker holds an entire Deck composed of fifty-two cards. After he has shuffled them, & after each Player or Punter has put a certain sum on a card taken at will, the Banker turns the deck, putting the bottom above; so that he sees the bottom card. Next he draws all his cards two by two until the end of the game, by commencing with the second. Here are the other rules of the game.

(1) The first card is for the Banker; but he takes only the two-thirds of the stake of the Punter when he brings forth his card, & this is called *facing*. The second is entirely for the Punter, the third entirely for the Banker, & thus in sequence alternately. It is necessary to remark that when a card has won or lost it no longer appears in the game, at least one does not replace it anew. Thus, for example, the card of the Punter being a King, if the first card of the deck is a Queen, the second a King & the third also a King, the Banker who says in drawing the cards, *King has won, King has lost* (this is understood of the Punter) will lose the stake of the Punter, although naturally the second King had made it winning, if the first card of the deal had been no King at all.

(2) When the Punters wish to take a card in the course of the game, it is necessary that the deal be low, that is to say that the Banker drawing them, as I have said two by two, has put the last deal or pair of cards under discussion, so that the card which remains revealed is losing for the Punters. Then if a Punter takes a card, the first card which the Banker will draw will be null in regard to this Punter, although it is favorable to the other Players; if it comes second, it will be faced, that is to say that the Banker will take $\frac{2}{3}$ of that which this Punter will have put on the card: if it comes in sequence, it will be in pure gain or in pure loss for the Banker, according as it will come, either first, or second from a deal.

(3) The last card, which must be for the Punter, is null.

PROPOSITION IX.
First Case.

We suppose that the Banker having six cards in his hands, the Punter takes one of them which is one time in these six cards, that is to say in the five covered cards. We ask what is the lot of the Banker with respect to this card of the Punter. For example, if the Punter puts an écu on his card, we ask to what part of the écu the advantage of the Banker is able to be evaluated.

Let the sought lot be expressed by S, & the stake of Paul by A.

If we imagine the one hundred-twenty different arrangements that five cards expressed by the letters a, b, c, d, f is able to receive, set under five columns, each of twenty-four arrangements; we will remark, (1) that that where the letter a occupies the first place, gives

A to the Banker. (2) That in each of the four other columns, the letter a is found six times in the third place, six times in the fourth, & six times in the fifth; whence it follows that we will have

$$S = \frac{24 \times A + 4 \times \overline{6 \times \frac{5}{3}A + 6 \times 2A + 6 \times 0 + 6 \times A}}{120} = \frac{136}{120}A = A + \frac{2}{15}A;$$

& consequently if A designates an écu worth sixty sols, Paul taking a card, under the conditions of the present Problem, would give to Pierre the same advantage as if he would give to him eight sols in pure gift.

We can next consider the other thing, by taking care that of these five columns, the first will give $24A$, the second $24 \times \frac{5}{3}A$, the third 24×0, the fourth $24 \times 2A$, & the fifth $24A$.

GENERALLY

If the card which the Punter takes is only one time among the covered cards of the Banker of which the number is expressed by p, we will have

$$S = \frac{3Ap + 2A}{3p}.$$

SECOND CASE

We suppose that the Banker holding six cards, the Punter takes one of them. Now as the card of the Punter is found either two times, or three times, or four times in these six cards, & as this diversifies the advantage of the Banker, it is proper to seek what is his lot in all the variations of this second case. I will begin by examining what is his lot under the supposition that the card of the Punter is two times in the hand of the Banker.

Let the five covered cards of the Banker be designated by the letters a, b, c, d, f, of which any two, for example, a, & f, express that of the Punter. We will remark, (1) that the one hundred-twenty different possible arrangements that the five cards can receive, being put under five columns each of twenty-four arrangements, of which the first begins with a, the second with b, the third with c, &c. the two columns which begin with a & with f give A to the Banker, because they are indifferent for the Banker & for the Punter. (2) That each of the three other columns contain twelve arrangements which give to the Banker $\frac{5}{3}A$, these are those where a & f are in the second place; & four arrangements which give $2A$ to the Banker, that is to say which make him win. This will be discovered easily from the Table here joined which represents the second column, which is that where b holds the first place.

$bacdf$	$bcadf$	$bdacf$	$bfadc$
$bacfd$	$bcafd$	$bdafc$	$bfacd$
$badcf$	$bcdaf$	$bdcaf$	$bfcad$
$badfc$	$bcdfa$	$bdcfa$	$bfcda$
$bafcd$	$bcfad$	$bdfac$	$bfdac$
$bafdc$	$bcfda$	$bdfca$	$bfdca$

It is clear that the first & the last of these four columns give $\frac{5}{3}A$ to the Banker, & that each of the two others contain two arrangements which give $2A$ to the Banker; these are these here, $bcdaf$, $bcdfa$, $bdcaf$, $bdcfa$. We will have therefore

$$S = \frac{2 \times 24A + 3 \times \overline{2 \times 6 \times \frac{5}{3}A + 2 \times 2 \times 2A}}{120} = \frac{11}{10}A = A + \frac{1}{10}A.$$

(2) In order to find what is the lot of the Banker when the card which the Punter takes is three times in the five cards of the Banker. We will observe that of the aforesaid five columns there are three which give A to the Banker, & two which contain each eighteen arrangements which give $\frac{5}{3}A$ to the Banker. This has no need of proof. We will have therefore

$$S = \frac{3 \times 24A + 2 \times 18 \times \frac{5}{3}A}{120} = \frac{132}{120}A = A + \frac{1}{10}A.$$

(3) In order to find what is the lot of the Banker when the card which the Punter takes is four times in the five cards covered by the Banker. We will observe that of the aforesaid five columns there are four which give A to the Banker, & one which gives to him $\frac{5}{3}A$. We will have therefore

$$S = \frac{4 \times 24A + 24 \times \frac{5}{3}A}{120} = A + \frac{2}{15}A.$$

THIRD CASE.

We suppose that the deck being composed of eight cards, of which the first is uncovered, the Punter takes one of them which is two times in these eight cards. We ask what is the lot of the Banker with respect to that card.

Let the seven covered cards be expressed by the seven letters a, b, c, d, f, g, h, of which two, namely a & f, designate that of the Punter. Let also, as above, S be the sought lot, & A the stake of Paul. This put,

We will observe, (1) that putting the five thousand forty different arrangements that the seven letters can receive on seven columns each of seven hundred twenty arrangements, the column which begins with a & that which begins with f, each will give A to the Banker. (2) That if we imagine each of the five others partitioned anew into six others of one hundred twenty arrangements each, the two from among these six where a & f occupy the second place, will give $\frac{5}{3}A$ to the Banker. (3) That the four other columns from among these six have each forty-eight arrangements which give $2A$ to the Banker. In order to see easily it is necessary to suppose that one of the five columns subdivided into six others, is that which begins with b, & to consult the Table which has served in the solution of the preceding case. We will remark first that the first and the last column of this Table being varied as much as it is possible with the two new letters g & h, a remaining in the second place, they will furnish each one hundred twenty arrangements which give $\frac{5}{3}A$ to the Banker. In regard to the four other columns of one hundred twenty arrangements each, in which the letters c, d, g, h would occupy the second place after b, it is easy to see that it suffices to examine one of them, since all four give the same lot to the Banker. Let the third column of the Table be that which we wish to examine. It is necessary to take care that each of the four arrangements $bcadf$, $bcafd$, $bcfad$, $bcfda$ being varied with the two new letters g & h, as much as it is possible, in such a way nonetheless that c remains in the second place, that is to say immediately after b, gives six new arrangements which make the Banker win, and give to him $2A$. For example, $bcadf$ furnishes these here,

$$\begin{array}{ll} bcgadfh & bchadfg \\ bcgadhf & bchagfd \\ bcgahdf & bchafgd \end{array}$$

It is thus of the three others of them, since g being in front of a or f, it is able to be found in three different places; & since h being in front of a or f, it is able to be found in three different places, a or f remaining always in the fourth.

We will find likewise that the two arrangements $bcdaf$, $bcdfa$ being varied as much as it is possible with g & h, in such a way nonetheless that c is always in the second place, furnish each twelve arrangements which give $2A$ to the Banker; because in $bcdaf$, g & h are able to be arranged in six ways with d, & in six different ways with f, a remaining in the fourth place; & likewise in $bcdfa$, g & h can be arranged in six ways with d, & in six different ways with a, f remaining always in the fourth place. From all this it follows that we will have

$$S = \frac{2 \times 720A + 5 \times \overline{2 \times 120 \times \frac{5}{3}A + 4 \times 48 \times 2A}}{5040} = \frac{536}{504}A = A + \frac{4}{63}A.$$

(2) In order to find what is the lot of the Banker when the card which the Punter takes is three times in the seven covered cards of the Banker.

Let the seven cards of the Banker be expressed as above by the letters a, b, c, d, f, g, h, of which any three, for example a, d, f, designate the card of the Punter. This put,

We will observe, (1) that putting the five thousand forty different arrangements that the seven cards can receive on seven columns of seven hundred twenty arrangements each, the three which begin with the letters a, d, f give A to the Banker, that which is evident. (2) That distributing each of the four others into seven columns of one hundred twenty arrangements each, the three columns from among these six where the letters a, d, f will take the second place, gives $\frac{5}{3}A$ to the Banker. (3) That each of the three other columns will contain thirty-six arrangements which will give $2A$ to the Banker. In order to be assured of this, one is able to consult the Table of page 45, & remark that each of the arrangements of the second column of the Table where b is in the first place, & c in the second, is able by the mixing of the two letters g & h, to receive only six arrangements which give $2A$ to the Banker, the first two remaining in their place. That which will appear evident, if we consider that in the six arrangements

$$\begin{array}{lll} bcadf & bcdaf & bcfad \\ bcafd & bcdfa & bcfda \end{array}$$

g or h being in front of one of the three letters a, d, f, h or g are able to be arranged in three different ways with the last two.

It is clear that it will be likewise of the three other columns of one hundred twenty arrangements where the first two letters could be bd, bg, bh. From all this it follows that we will have

$$S = \frac{3 \times 720A + 4 \times 3 \times \overline{3 \times 120 \times \frac{5}{3}A + 3 \times 36 \times 2A}}{5040} = A + \frac{8}{105}A.$$

(3) In order to find what is the lot of the Banker when the deck being composed of seven covered cards, the Punter takes one of them which is four times in these seven cards; we will observe, (1) that imagining the five thousand forty possible arrangements of seven cards put on seven columns of seven hundred twenty arrangements each, of which the one

begins with a, the second with b, &c. as above, there will be four of these seven which will give A to the Banker. (2) By distributing each of the three others on six columns of one hundred twenty arrangements each, four of these six will furnish each one hundred twenty arrangements which will give $\frac{5}{3}A$ to the Banker, & the two others twenty-four arrangements each which will give to him $2A$. We will have therefore

$$S = \frac{4 \times 720A + 3 \times \overline{4 \times 120 \times \frac{5}{3}A + 2 \times 24 \times 2A}}{5040} = A + \frac{11}{105}A$$

It would be useless to pursue in detail the solution of a greater number of cases. We see rather by the preceding reflections, what will be those which would be necessary to make under the assumption that the Banker having nine covered cards, the Punter takes one of them. Thus, (1) we will find that if the card of the Punter is two times in these nine cards, we will have

$$S = \frac{2 \times 40320A + 7 \times \overline{2 \times 5040 \times \frac{5}{3}A + 6 \times 2160 \times 2A}}{5040 \times 8 \times 9} = \frac{379680}{362880}A = A + \frac{5}{108}A.$$

(2) If the card of the Punter is three times in the nine cards of the Banker, we will have

$$S = \frac{3 \times 40320A + 6 \times \overline{3 \times 5040 \times \frac{5}{3}A + 5 \times 9360 \times 2A}}{362880} = \frac{284480}{362880}A = A + \frac{5}{84}A.$$

(3) Finally if the card of the Punter is four times in these nine cards, we will have

$$S = \frac{4 \times 40320A + 5 \times \overline{4 \times 5040 \times \frac{5}{3}A + 4 \times 1584 \times 2A}}{362880} = \frac{392640}{362880}A = A + \frac{31}{378}A.$$

One will find easily in the preceding reflections & in the order of the original numbers & the demonstration of the formulas $B = \frac{1}{3} \times \frac{p+1}{pp-p}$, $C = \frac{pp-2p-3}{2p^3-6pp+4p}$, $D = \frac{2pp-3p-11}{3p^3-9pp+6p}$, in which p designating the number of covered cards that the Banker holds, B expresses his advantage when the card of the Puner is found twice in the stock, C his advantage when it is there three times, D his advantage when it it there four times.

These formulas are convenient enough in order to calculate it, but they are particular & limited; here is two other of them very universal, which would hold, for example, if the Banker dealt with two or three decks of cards mixed together.

GENERALLY.

Whatever number of cards that the Banker holds, & whatever number of times that that of the Punter is contained there, if one names p the number of covered cards that the Banker holds; q the number of times that that of the Punter is contained there, one will have the lot of the Banker expressed by this general formula,

$$\begin{aligned}
S \;=\; & \frac{q}{p}A + \frac{q \times p-q}{p \times p-1} \times \frac{5}{3}A + \frac{q \times p-q \times p-q-1 \times p-q-2}{p \times p-1 \times p-2 \times p-3} \times 2A \\
+ \; & \frac{q \times p-q \times p-q-1 \times p-q-2 \times p-q-3 \times p-q-4}{p \times p-1 \times p-2 \times p-3 \times p-4 \times p-5} \times 2A \\
+ \; & \frac{q \times p-q \times p-q-1 \times p-q-2 \times p-q-3 \times p-q-4 \times p-q-5 \times p-q-6}{p \times p-1 \times p-2 \times p-3 \times p-4 \times p-5 \times p-6 \times p-7} \times 2A \\
+ \; & \frac{\begin{array}{c} q \times p-q \times p-q-1 \times p-q-2 \times p-q-3 \times p-q-4 \\ \times p-q-5 \times p-q-6 \times p-q-7 \times p-q-8 \end{array}}{p \times p-1 \times p-2 \times p-3 \times p-4 \times p-5 \times p-6 \times p-7 \times p-8 \times p-9} \times 2A \\
+ \; & \text{&c.}
\end{aligned}$$

It is easy to continue this formula, & to discover all the terms of this series, by commencing from the third.

In order to have the lot of the Banker one will employ all the terms of this series until this that one comes to a term of it which is $= 0$; but it is necessary to observe that when the card of the Punter is only once in the stock, it is necessary to add $\frac{q}{p}A$ to the terms of the series.

ANOTHER FORMULA

If one names g the lot of the Banker in a number of cards expressed by $p-2$, & the rest as above, one will have generally

$$\begin{aligned} S &= \frac{q}{p}A + \frac{p-q}{p} \times \frac{q}{p-1} \times \frac{5}{3}A + \frac{q \times p-q \times p-q-1 \times p-q-2}{p \times p-1 \times p-2 \times p-3} \times 2A \\ &+ \frac{p-q \times p-q-1}{p \times p-1} \times \overline{g-1 \times \frac{q}{p-2}A - \frac{p-q-2}{p-2 \times p-3} \times q \times \frac{5}{3}A}. \end{aligned}$$

REMARK I.

In this Game, as in the one of Pharaon, the greatest advantage of the Banker is when the Punter takes one card which has not passed at all, & his least advantage is when the Punter take one of them which has passed twice; his advantage is also greater when the card of the Punter has passed three times, than when it has passed only one time.

REMARK II.

In the Game of Bassete the advantage of the Banker is less than in the game of Pharaon, that which we will understand easily by comparing the advantage of the Banker in the game of Bassete, when taking twelve cards the Punter takes one of them which is found either one, or two, or three, or four times, with his lot in this same case in the Game of Pharaon.

We will find that the Punter setting a pistole on his card at Bassete, the advantage of the Banker will be 13 f. 4 d. when the card of the Punter will be four times in the twelve cards of the Banker, 12 f. 1 d. when it will be one time, 9 f. 8 d. when it will be three times, & 7 f. 3 d. when it will be twice; instead that in Pharaon the advantage is 19 f. 2 d. $\frac{10}{33}$ in the first case, 16 f. 8 d. in the second, 13 f. $7\frac{7}{11}$ d. in the third, & 10 f. $7\frac{3}{11}$ d. in the fourth, that which gives 3 liv. 1 denier advantage to the Banker for the four cases; instead that in Bassete the four together give only 2 liv. 2 f. 4 den. that which is just slightly less than two-thirds of the advantage of the Banker in the game of Pharaon.

REMARK III.

This game is presently much less in use than Pharaon. The cards which do not go, make losing in the game something of its vivacity. Besides there are often some disputes for knowing if the card of the Punter goes or does not go. We can not remedy these inconveniences, which are based on the nature of the Game; but we would render this game more equal by agreeing that the faced cards paid only the half of the stake of the Punter, then the advantage of the Banker would be much less great, I have found that if the Banker took only a third for the faces, this game would be disadvantageous to him. The greater part of the Remarks which we have made on the game of Pharaon, are able to hold in regard to this one, & it will not be useless to consult them.

TABLE FOR BASSETE

5	$1 := a + \frac{2}{15}a$	$2 := a + \frac{1}{10}a$	$3 := a + \frac{1}{10}a$	$4 := a + \frac{2}{15}a$
7	$1 := a + \frac{2}{21}a$	$2 := a + \frac{4}{63}a$	$3 := a + \frac{8}{105}a$	$4 := a + \frac{11}{105}a$
9	$1 := a + \frac{2}{27}a$	$2 := a + \frac{5}{108}a$	$3 := a + \frac{5}{84}a$	$4 := a + \frac{31}{378}a$
11	$1 := a + \frac{2}{33}a$	$2 := a + \frac{2}{55}a$	$3 := a + \frac{8}{165}a$	$4 := a + \frac{11}{165}a$
13	$1 := a + \frac{2}{39}a$	$2 := a + \frac{7}{234}a$	$3 := a + \frac{35}{858}a$	$4 := a + \frac{40}{715}a$
15	$1 := a + \frac{2}{45}a$	$2 := a + \frac{8}{315}a$	$3 := a + \frac{16}{455}a$	$4 := a + \frac{197}{4095}a$
17	$1 := a + \frac{2}{51}a$	$2 := a + \frac{3}{136}a$	$3 := a + \frac{21}{680}a$	$4 := a + \frac{129}{3060}a$
19	$1 := a + \frac{2}{57}a$	$2 := a + \frac{10}{513}a$	$3 := a + \frac{80}{2907}a$	$4 := a + \frac{109}{2907}a$
21	$1 := a + \frac{2}{63}a$	$2 := a + \frac{11}{630}a$	$3 := a + \frac{33}{1330}a$	$4 := a + \frac{202}{5985}a$
23	$1 := a + \frac{2}{69}a$	$2 := a + \frac{12}{759}a$	$3 := a + \frac{120}{5313}$	$4 := a + \frac{163}{5313}a$
25	$1 := a + \frac{2}{75}a$	$2 := a + \frac{13}{900}a$	$3 := a + \frac{153}{6900}a$	$4 := a + \frac{291}{10350}a$
27	$1 := a + \frac{2}{81}a$	$2 := a + \frac{14}{1053}a$	$3 := a + \frac{56}{2925}a$	$4 := a + \frac{633}{26325}a$
29	$1 := a + \frac{2}{87}a$	$2 := a + \frac{15}{1218}a$	$3 := a + \frac{195}{10962}a$	$4 := a + \frac{44}{1827}a$
31	$1 := a + \frac{2}{93}a$	$2 := a + \frac{16}{1395}a$	$3 := a + \frac{224}{13485}a$	$4 := a + \frac{101}{4495}a$
33	$1 := a + \frac{2}{99}a$	$2 := a + \frac{17}{1584}a$	$3 := a + \frac{85}{5456}a$	$4 := a + \frac{517}{24552}a$
35	$1 := a + \frac{2}{105}a$	$2 := a + \frac{18}{1785}a$	$3 := a + \frac{288}{19635}a$	$4 := a + \frac{389}{19635}a$
37	$1 := a + \frac{2}{111}a$	$2 := a + \frac{19}{1998}a$	$3 := a + \frac{323}{23310}a$	$4 := a + \frac{218}{11655}a$
39	$1 := a + \frac{2}{117}a$	$2 := a + \frac{20}{2223}a$	$3 := a + \frac{120}{9139}a$	$4 := a + \frac{1457}{82251}a$
41	$1 := a + \frac{2}{123}a$	$2 := a + \frac{21}{2460}a$	$3 := a + \frac{399}{31980}a$	$4 := a + \frac{269}{15990}a$
43	$1 := a + \frac{2}{129}a$	$2 := a + \frac{22}{2709}a$	$3 := a + \frac{440}{37023}a$	$4 := a + \frac{593}{37023}a$
45	$1 := a + \frac{2}{135}a$	$2 := a + \frac{23}{2970}a$	$3 := a + \frac{161}{14190}a$	$4 := a + \frac{976}{63855}a$
47	$1 := a + \frac{2}{141}a$	$2 := a + \frac{24}{3243}a$	$3 := a + \frac{528}{48645}a$	$4 := a + \frac{711}{48645}a$
49	$1 := a + \frac{2}{147}a$	$2 := a + \frac{25}{3528}a$	$3 := a + \frac{575}{55272}a$	$4 := a + \frac{129}{9212}a$
51	$1 := * \quad * \quad *$	$2 := * \quad * \quad *$	$3 := a + \frac{208}{20825}a$	$4 := a + \frac{2519}{187425}a$
52	$1 := * \quad * \quad *$	$2 := * \quad * \quad *$	$3 := * \quad * \quad *$	$4 := a + \frac{2453842}{175592235}a$

PROBLEMS

ON PIQUET, HOMBRE, TRUMP, BRELAN, AND IMPERIAL

FOREWARD

When chance rules absolutely in a game, one is always able to determine the advantage or the disadvantage of the players; the preceding Problems are able to serve as proof of it; & if one pays attention to the variety of the conditions of these games, & to the great number of circumstances to which it has been necessary to have regard, one will recognize that the greater part of the other games of pure chance, that one knows or that one is able to imagine, will be determined by some methods either similar, or little different from those which have served to resolve the preceding Problems.

It is not likewise of the games where the knowledge of the player has part in the event as well as the lot, because this knowledge, which does not merit the name, being based only on some misleading rules of possibility, & most often on the caprice & the fantasy of the players, it is impossible that the conjectures that one forms on these rules, not participate in their uncertainty. Thus the light that has led us to here in the games of pure chance, must be lacking to us in the greater part of the questions that one is able to make on the games of which the good or bad events for the players, depend not at all entirely on fortune. It is proper to clarify & to prove this here by some examples.

A Problem that one proposes often on Piquet, it is to know how much between two equal players, a first in card is able to wager to make points. One believes commonly that this is able to go to twenty-eight points, & it is on this basis that I have seen the part of it done by some good players. Now in order that a first in card could resolve this question, it would be necessary that he knew not only the number of different dispositions that his twelve cards are able to receive, & those of the last, & that he knew further the art of comparing all the changes which are able to arrive to his twelve cards when he will set aside five of them in order to take so many of them in the stock, & to the twelve cards of the last when he will set aside three of them in order to take three of them in the stock. It would be further necessary that he knew that which the last must set aside in each of the different possible dispositions of his twelve cards. Now it is there that which the first is not able to know, the last not knowing it himself, because he does not play at all who has some fixed & certain rules for all the possible different dispositions of the game. Nonetheless without this last knowledge, the first is nearly useless to him who is first in cards, & there would never be able to be made some sure rules in order to set aside to that purpose, & next in order to play the cards well.

Let us suppose further that a player wishes to examine that which is most advantageous to him to set aside, a major quarte or a quarte of King. It is true that he will perceive without difficulty that by guarding the quarte of King, there are two cards which are able to give to him a quinte, against one if he guards the major quarte; but he would not know how to conclude what part he must take, because beyond that which depends on the state where the game is, it is necessary that he have regard to the disposition of the rest of his game, that he consider that which he has to fear from his adversary, he must think to make the cards or to render them equals, &c. Now all that demands a great number of comparisons of which each would be the material of a quite composed Problem. Thus it is necessary to swear that in the examination of the detail of this game, the theory is not able to lead very far.

The first rule of Analysis, it is that one is able to discover that which is unknown only by means of that which is known. Now in the two preceding questions that which is known is not sufficient in order to discover that which is to find.

It is thus of the greater part of the questions that one is able to propose on the game of Hombre, & so much more, when one plays three, with forty cards, & when there remains a greater number of cards in the stock. This is why in the greatest part of the difficulties that present themselves with respect to this game, it is necessary to be content to seek the possible, & to limit its study to approach the truth as much as it is possible. However well some players understand the art of guessing much better than me, I would not abandon showing by example that follows, in what manner it is necessary to take.

Let be supposed that Pierre has made to play in spade, that he has four hands, & that playing his fifth there remains to him yet two sure trumps, & beyond that the King of diamonds & the Queen of hearts. One demands if Pierre must hold to take all tricks.

In order to resolve exactly this Problem, it would be necessary to make a thousand circumstances enter of which one could be able to calculate the precise value only with a very great labor; but if one wishes to be content with the possible, it will suffice to observe what are the principal encounters where Pierre undertaking the slam lost, what are those which would render him certain to win, & what are those also which would render success uncertain. Thus in the present case one will note that Pierre will win, if the King of clubs being in one hand, the King of hearts is in the other hand with protection in diamonds, now if the two Kings being in one same hand with protection in diamonds, this protection is not in the other hand at all, or is less advantageous.

(2) That Pierre will lose if none of the two players having protection in diamonds, the two Kings are in a different hand, or if one of the two players has protection in diamonds & the King of clubs, the other player having the King of hearts without protection in diamonds, or with a protection less advantageous than that which accompanies the King of clubs.

(3) That if the two Kings are found in one same hand without that any of the two players has protection in diamonds, there will be for Pierre so much expectation to win as risk of loss.

One will be able in weighing these reasons for & against, & making some other circumstances enter, for example that here, that protection in diamonds is able to be so low that the player will be determined rather to guard his King than this covering card; one will be able, I say, by examining how much one of these cases furnish more encounters than another, to take from this comparison some quite possible reasons in order to be determined. For me I swear that I would prefer to take all the tricks; & although apparently that has been practiced by nobody, I am persuaded that those who pay attention to that which precedes, would not be very far from my sentiment; there is presented very often with difficulties of this nature, & these are so many Problems that it is necessary to resolve, & resolve immediately. This is why it is necessary to agree that a man who has the lively & penetrating mind, & who has the practice of the game, has much more advantage to well take his part in the greater part of the encounters of this game, than another player who with so much practice will have the imagination less just & less acting, because it is not necessary less wit in order to encounter the possible when the evidence is lacking, than in order to discover the truth when it is possible to find it.

Brelan, & generally all the games where one raises the stakes are subjects to the same inconveniences as the game of Hombre, & even some greater. Let us suppose for example, that there are three players, Pierre, Paul & Jacques; Pierre passes, Paul persists in playing, & Jacques raises the stakes; Paul keeps the stake, & goes from all that which he has before him, this will be for example $30A$, the game being A. One demands if Jacques, who one supposes to have forty-one in hand, & who is last, must hold or abandon that which he has already set into the game, for example $14A$. I know that well some persons would not hesitage to decide the above for or against, each consulting his mood rather than the evidence. For me I believe to be able to be assured that it is impossible to determine exactly what part Jacques must take, & my reason is that it does not suffice to Jacques in order to determine with reason, to know which among 134596 different ways of which the cards of Pierre & Paul are able to be disposed, there are only 3041 of them which are able to make Jacques lose. It would be necessary that there were certain & known rules to the two Players in order to know at what card it is necessary to keep the game, & until when it is proper to keep or to push for each game. Then Jacques would be able to count that Paul has one of the games which have been able to permit him to go all, & on that it would be able very nearly to be determined; I say very nearly, because it would not be sure that Paul in order to give the change to him, not push to a game quite inferior to the one that he ought have in order to force with reason, & thence Jacques would be exposed to lacking to win, & even to lose his advantage when he ought to win.

These reflections & some others parallel that everyone is able to make, are sufficient in order to make known that there are in these matters some Problems that it is impossible to resolve, & that one must not at all consequently await to find in this Book. The following examples will make known in what nature are those of which the research is able to be tried in these matters with expectation of success. I have put only a small number of them, & I have chosen among those which have appeared to me curious & of some use for the Players, those which I have believed most proper to make known the use of the lemmas which follow. It will be easy to perceive that one is able to apply them to some more important researches than ours are.

DEFINITION

One understands sometimes by this term combination, the manner by which many things are able to be taken differently two by two. I will give to it here a more extended signification, & I will understand by this word the manner to find generally all the dispositions that either two, or many things are able to have according as one will wish to take them, either two by two, or three by three, or four by four, or five by five, or finally in all possible manners.

PROBLEM
PROPOSITION X

Any number of things whatsoever being proposed, for example the letters a, b, c, d, f, g, h, *&c. one demands how many different ways there are to take them, either one by one, or two by two, or three by three, or finally in all the ways possible.*

In order to resolve this Problem, I will serve myself with the following Table joined here, of which I am going to explicate the formation, & of which I will demonstrate next the usage with respect to combinations.

Table of Mr. Pascal for the combinations.

1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13
		1	3	6	10	15	21	28	36	45	55	66	78
			1	4	10	20	35	56	84	120	165	220	286
				1	5	15	35	70	126	210	330	495	715
					1	6	21	56	126	252	462	792	1287
						1	7	28	84	210	462	924	1716
							1	8	36	120	330	792	1716
								1	9	45	165	495	1287
									1	10	55	220	715
										1	11	66	286
											1	12	78
												1	13
													1

I call horizontal bands those where the numbers go from left to right, & perpendicular bands those where the numbers go from high to low; I call cell the position of a number contained between two points.

The second horizontal band is the sequence of natural numbers, one, two, three, four, &c.

The third horizontal band is formed out of the second in this manner; (1) I go back from left to right by one cell: (2) in order to form the number of each cell of this band, I add all the numbers which precede it to left in the superior horizontal band. Thus the number six, third number of the third horizontal band, is equal to the sum of the first, of the second & of the third number of the second horizontal band.

The fourth horizontal band is formed out of the third in the same manner as the third is formed out of the second: Thus one will find that the number 20 which is the fourth of the fourth horizontal band, is equal to the sum of the four numbers which precede it in the superior horizontal band which is the third. It would be likewise of all the other numbers of this fourth band.

One will form the numbers which compose the other horizontal bands in the same manner as one has formed the second out of the first, & the third out of the second, observing always to go back each band from a cell advancing toward the right; it is that which one will be able to discover easily by considering the table that one will be able to continue to infinity.

The numbers which compose the first horizontal band are called numbers of the first order, those which compose the second horizontal band are called numbers of the second order, those which compose the third band are called numbers of the third order, &c.

These numbers to which one gives also the names of units, of natural numbers, triangular numbers, pyramidals, triangulo-pyramidals, &c. because of certain relations that they have to the triangles, to the Pyramids, &c. have some properties quite singular, that Messrs. Fermat, Descartes, Pascal, & many other great French & Foreign Geometers have researched with great care. One of the principals, & of which there is concern here, is that by their means one is able to find all at once in how many different ways any number of tokens or of cards or of each other thing, is able to be combined, that is taken either one by one, two by two, or three by three, or four by four, &c. in a great number of tokens & cards.

For example if one demands in how many different ways six different things are able to be taken two by two; one will find that the number fifteen which corresponds to the third horizontal band & to the seventh perpendicular band, is the number that one seeks: & likewise if one wishes to know in how many different ways eleven things are able to be taken four by four, one will find that the number 330 which corresponds to the fifth horizontal band & the twelfth perpendicular band, is the number that one demands. One will find likewise all the other imaginable combinations by seeking the number which corresponds to a perpendicular column of which the number surpasses by unity the number of things proposed, & to a horizontal column which is the third if the things are combined two by two, the fourth if the things are combined three by three, &c.

Mr. Pascal is the first who has discovered this usage of the numbers of different orders, & one is able to see the demonstration in the Treatise that he a made entitled *Triangle Arithmetique*, where he applied these numbers so much to the combinations, as to find the divisions that two Players must make who playing to a certain number of points in an equal game, have more or less points.

DEMONSTRATION.

In order for me to be more easily understood, I take an example, & I suppose that one wishes to know in how many different ways six things are able to be taken either one by one, or two by two, or three by three, or four by four, or five by five, or six by six, let these six things whatsoever be expressed by the six letters a, b, c, d, f, g.

Firstly it is evident that if one seeks in how many ways these six letters are able to be taken one by one, the number six will be the one which satisfied the Problem. Now it is evident that the terms of the first horizontal band which precedes the number six of the second, being added to a sum make the number six.

Let us suppose next that one wishes to know in how many different ways these same letters are able to be taken two by two. In order to find it one will observe, (1) that the letter a is able to be combined with the five following b, c, d, f, g. (2) That the letter b is able to be combined differently with the four letters following c, d, f, g, that which gives four different combinations bc, bd, bg, bf; for ba will make quite a different arrangement than ab; but not a different combination. (3) That c is combined only with the letters d, g, f; for ca, cb, would not be at all different combinations. (4) That d is combined only with the two letters f & g; because da, dc, db, would not be at all different combinations. (5) That f is combined only one time with g; because fa, fb, fc, fd, would be repetitions of the preceding combinations, that which it is necessary to observe with care; because it is the fundamental principal of the demonstration.

All these combinations together of six letters taken two by two, are

$$\begin{array}{ccccc} ab, & ac, & ad, & ag, & af \\ & bc, & bd, & bg, & bf \\ & & cd, & cg, & cf \\ & & & dg, & df \\ & & & & fg \end{array}$$

of which the sum $5 + 4 + 3 + 2 + 1 = 15$.

And consequently the number 15 which is found in the seventh perpendicular band & in the third horizontal band, is the sum of the numbers which precede it to left in the superior horizontal band, & is at the same time the number which expresses in how many different ways six letters are able to be taken two by two.

Let us suppose now that one wishes to find in how many different ways these six letters are able to be taken three by three.

One will note (1) that ab is able to be combined in four ways with the letters c, d, f, g; ac in three ways, ad in two ways, & af only in one way.

(2) That bc is combined in three ways with the letters d, f, g; bd in two ways with the letters f & g, & bf only in one way with g.

(3) That cd is combined in two ways with the lettes f & g, & cf only in one way with g.

(4) It is evident that df is able to be combined only in one way with g. All these combinations together of six things taken three by three, are

$$\begin{array}{cccccccccc} abc, & abd, & abf, & abg, & acd, & acf, & acg, & adf, & adg, & afg \\ & & & & bcd, & bcf, & bcg, & bdf, & bdg, & bfg \\ & & & & & & & cdf, & cdg, & cfg \\ & & & & & & & & & dfg \end{array}$$

of which the sum $10 + 6 + 3 + 1 = 10$.

And consequently the number 10 which is found in the seventh perpendicular band, & in the fourth horizontal band, is the sum of the numbers which precede it to left in the superior horizontal band, & is at the same time the number which expresses in how many different ways six letters are able to be taken three by three. Therefore, &c.

Let us suppose next that one wishes to know in how many different ways these six letters are able to be taken four by four.

One will observe (1) that abc is able to be combined in three different ways with the letters d, f, g; abd in two ways with f & g; abf only with g; that acd is able to be combined in two ways with the letters f & g; that acf & adf are combined only in one way.

(2) That bcd is combined in two ways with the letters f & g, & that bcf, bdf & cdf are combined only in one way with g.

All these combinations together are

$$\begin{array}{cccccccccc} abcd, & abcf, & abcg, & abdf, & abdg, & abfg, & acdf, & acdg, & acfg, & adfg \\ & & & & & & bcdf, & bcdg, & bcfg, & bdfg \\ & & & & & & & & & cdfg \end{array}$$

of which the sum $10 + 4 + 1 = 15$.

And consequently the number 15 which is in the seventh perpendicular band & in the fifth horizontal band, is the sum of the numbers which precede it to left in the superior band, & is at the same time the number which expresses in how many different ways six letters are able to be taken four by four.

If one wishes further to know in how many different ways these six letters are able to be taken five by five, one will note (1) that $abcd$ is able to be combined differently only with the two letters f & g, $abcf$ only in one way with g, $abdf$ in only one way with g; & $acdf$ only on one way with g. (2) That $bcdf$ is combined only in one way with g.

The sum of these combinations of six things taken five by five, will be therefore $abcdf$, $abcdg$, $abcfg$, $abdfg$, $acdfg$, $bcdfg$, $= 6$.

And consequently the number 6 which is in the seventh perpendicular band & in the sixth horizontal band, is the sum of the numbers which precede it in the superior horizontal band, which is that of the numbers of the fifth order; & is at the same time the number which expresses in how many different ways six letters are able to be taken five by five.

Finally it is evident that six letters are able to be taken in one way six by six.

From all that it is necessary to conclude that the seventh perpendicular column expresses all the possible ways, of which six things are able to be taken, either one by one, or two by two, or three by three, or four by four, or five by five, or six by six.

One will find likewise that the eighth perpendicular column expresses all the possible ways in which seven things are able to be taken, either one by one, or two by two, or three by three, or four by four, or five by five, &c. And finally that this table being continued to infinity, would give all the possible ways of which any number of tokens or of cards would be able to be taken either one by one, or two by two, or three by three, &c. in a greater number of tokens or of cards. *That which it was necessary to demonstrate.*

One draws from the preceding demonstration an easy & short way to form the table, it is to know to add into one sum the number which precedes the sought number to left in the same horizontal band, & the number which is superior to the one which is to the left; thus in order to form the third horizontal band, I add the number which is to the left (it is zero) & the number above, that gives me one for the first term of this band. In order to have the second, I add the number 1 which is to the left of the sought number with the number 2 which is superior to it, the sum $2 + 1 = 3$ will be the second of the third horizontal band; the third term of this band will be $3 + 3 = 6$; the fourth will be $6 + 4 = 10$, & thus consecutively. If one wishes for example to find the number which corresponds to the ninth perpendicular band & to the sixth horizontal band, I add 35 to 21, the sum which is 56 is the sought number.

This manner to consider the formation of this table, presents a new demonstration of its usage for the combinations, which is simpler & shorter than the preceding.

Another Demonstration.

Let it be supposed that one wishes to take five things for example in all possible ways, either two by two, or three by three, or four by four, or five by five. It is clear (1) that five things a, b, c, d, e are able to be taken two by two in as many ways as four things have been taken in this manner, (or the letters a, b, c, d, have been able to be set two by two in six ways, namely ab, ac, ad, bc, bd, cd) & that they are able to be taken beyond that in as many ways as four things are able to be taken one by one, namely ae, be, ce, de, that which gives ten combinations of five things taken two by two. (2) Five things are able to be taken three by three in as many ways as four things have been taken three by three & two by two. Now four things are able to be taken three by three in four ways, abc, acb, abd, bcd; & two by two in six ways; ab, ac, ad, bc, bd, cd; therefore if one adds the letter e to these last six different ways, one will find that the number 10 expresses in how many different ways five things are able to be taken three by three, & is at the same time the sum of the number which precedes it to left, & the one which is above this number. This demonstration extends to all the numbers of the table, & is based on this that any number p is able to be taken in any other number, but greater, q in as many ways as p & $p - 1$ are able to be taken in $q - 1$. Now this proposition is evident in regard to the number which is to left, since the small is contained in the greatest; it is true also in regard to the one which is superior to the number at the left, since by joining the letter which is not entered into the combinations that the number expresses to the left, it furnished it anew, & supplied to those which are lacking to the number at the left. Therefore, &c.

REMARK.

In order to spare the difficulty to the Reader to form some tables which are able to serve to find all the combinations of which one will have need in the following, that which is of excessive length when the combinations that one seeks are among great numbers. For example, when one of the numbers being 49, the other is 100, it is useful & even necessary to find some formula which is able to give the sought number without having need to know all the possible combinations among the lesser numbers. It is that which one will learn by the Lemma which follows.

PROPOSITION XI.
LEMMA

In order to find all at once such term as one will wish of any perpendicular column of the table page 54 continued at discretion, one will multiply the one which is immediately above by the exponent of the perpendicular band, less the exponent of the horizontal band plus one. This product divided by the exponent of the horizontal column less one will give the sought number; so that naming, for example the first term of any perpendicular column 1, the second p*, the third* B*, the fourth* C*, the fifth* D*, &c. one will have the first* $= 1$*, the second* $p = 1 \times \frac{p+1-2+1}{2-1}$*, the third* $B = p \times \frac{p+1-3+1}{3-1} = p \times \frac{p-1}{2}$*, the fourth* $C = B \times \frac{p+1-4+1}{4-1} = B \times \frac{p-2}{3} = p \times \frac{p-1}{2} \times \frac{p-2}{3}$*, the fifth* $D = C \times \frac{p+1-5+1}{5-1} = C \times \frac{p-3}{4} = p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$*, &c.*

In order to demonstrate this Lemma, I will make first to perceive the truth on one of the perpendicular columns of the table taken at discretion, & after this induction I will prove that all the others to infinity must follow this same law.

It is clear by examining, for example, the six where $p = 5$, that one will have $B = \frac{5\times4}{2} = 10$, $C = \frac{10\times3}{5} = 10$, $D = \frac{10\times2}{4} = 5$, $E = \frac{5\times1}{5} = 1$; & that thus by following the rule of the Lemma, one finds each term of this column such as the formation of the table gives it.

It is necessary now to prove that this rule holding as by chance for this column, it holds by necessity in regard to the others.

Let the seventh column be that one wishes to examine, & of which one must find by the Lemma all the terms conformed to those that the formation of the table gives.

It is clear that the first of this band will be 1, & that the second will be $p + 1$, one will find by the Lemma that the third is $\frac{p+1\times p}{2} = 15$, & by the formation of the table that it is $p \times \frac{p-1}{2} + p = \frac{pp-p+2p}{2} = \frac{pp+p}{2}$, that which it is necessary firstly to find. If one supposes now that $p + 1$ of the quantity that one has just found is changed into p, one will find by substituting p for $p + 1$ into the quantity $p \times \frac{p+1}{2}$ that it is changed into this here $p \times \frac{p-1}{2}$ whence if follows that one will find by this means the third term of the eighth perpendicular band $= p \times \frac{p+1}{2} = 21$ & that employing always the same artifice, one would find so forth conformably to the Lemma all the terms of the third horizontal band, such as the formation of the table gives them. One will find also by the Lemma the 4^{th} term of this seventh horizontal band $= \frac{p+1\times p}{2} \times \frac{p-1}{3} = \frac{p^2-p}{6} = 20$, & by the formation this other $\frac{p\times p-1}{2} \times \frac{p-2}{3} + \frac{p\times p-1}{2} = \frac{p^3-3pp+2p+3pp-3p}{1\times2\times3} = \frac{p^3-p}{6}$ which is equal to it. Now if one substitutes into this quantity, p for $p + 1$, it will be changed into this here, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, & by virtue of this supposition one would find the fourth term of the eighth perpendicular band $= \frac{p^3-p}{6} = 35$, & thus one will be assured by the truth of the Lemma in regard to all the terms of this fourth horizontal band.

One will find further by the Lemma the fifth term of the seventh perpendicular band $= \frac{\frac{p^3-p\times p-2}{6}}{4} = \frac{p^4-2p^3-pp+2p}{24} = 15$, & by the formation the same term under this form $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} + p \times \frac{p-1}{2} \times \frac{p-2}{3}$ which is reduced to this here $\frac{p^4-2p^3-pp+2p}{24}$.

Now if one substitutes into this quantity, p in the place of $p+1$, it will be changed into this here, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$, which will serve to find all the terms of the fifth horizontal band, thus as one has taught it before.

One will find finally by the Lemma the sixth term of the seventh perpendicular band $= \frac{\frac{p^4-2p^3-pp+2p}{24}\times p-3}{5} = \frac{p^5-5p^4+5p^3+5pp-6p}{120} = 6$, & by the formation this same term $= \frac{p\times p-1\times p-2\times p-3\times p-3\times p-4}{1\times 2\times 3\times 4\times 5} + \frac{p\times p-1\times p-2\times p-3}{1\times 2\times 3\times 4} = \frac{p^5-5p^4+5p^3+5pp-6p}{120}$; & by substituting as one has done before p in the place of $p+1$, one will be assured that the following term & all the others of this sixth horizontal column must follow the order that the Theorem teaches.

COROLLARY I.

It follows from the preceding Lemma that if one seeks in how many ways the number q is able to be taken in another greater number which is called p, the sought number will be expressed by a fraction of which the numerator will be equal to as many products of p, $p-1$, $p-2$, $p-3$, $p-4$, &c. as q expresses units, & of which the denominator will be composed of an equal number of products of the natural numbers 1, 2, 3, 4, 5, 6, &c.

COROLLARY II.

If one wishes to take either p or q in a number expressed by m, I say that if $p+q=m$, the number which will express in how many ways one is able to take p in m will be the same as the one which expresses in how many ways one is able to take q. Thus for example m being 7, the number which will express in how many ways one is able to take three things in seven, will be the same as the one which expresses in how many ways one is able to take in it four; & likewise the number which will express in how many ways one is able to take two things in seven, will be the same as the one which expresses in how many ways one is able to take five in it; & the number which will express in how many ways one is able to take one thing in seven, will express in how many ways one is able to take six. It follows thence (1) that if m expresses an odd number, the two numbers of the perpendicular column which are the most distant from the extremities, are equal & the greatest among all those of the column; & that if m expresses an even number, the one of the middle will be the greatest between all the numbers of this column. (2) That the numbers which are at equal distance, either from the one of the middle if m is an even number, or from the middle two if it is odd, will be equal to one another. (3) One is able to observe that the sum of all the terms of any perpendicular band is equal to the corresponding term of a double geometric progression, of which the first term is unity.

Thus for example one will find that the eighth term of a double geometric progression which is 128, will be equal to the sum of all the numbers which contain the eighth perpendicular band.

COROLLARY III.

In all the particular applications, the formula of the first Corollary is always reduced to a whole number; thus, for example, $\frac{p\times p-1\times p-2\times p-3\times p-4\times p-5\times p-6\times p-7\times p-8\times p-9}{1.2.3.4.5.6.7.8.9.10}$ &c. the formula which, in supposing $p=40$, would express in how many ways ten things are able to be taken in 40, is reduced by dividing the numerator & the denominator as much as it is possible, to the quantity $31\times 4\times 11\times 34\times 37\times 38\times 13 = 847660528$.

COROLLARY IV.

The number p expressing the series of natural numbers, the formula of the triangular will be $p \times \frac{p+1}{2}$, that of the pyramidal will be $p \times \frac{p+1}{2} \times \frac{p+2}{3}$, that of the triangulo-pyramidal will be $p \times \frac{p+1}{2} \times \frac{p+2}{3} \times \frac{p+3}{4}$, &c. so that if one ranks the numbers of different orders in such manner as not going back from a number, as in the table page 54, from a superior horizontal band to the following, they make a square figure in the place of the triangular figure that they have in the table 54, if one wishes to find in this new table such figured number as one will wish, its order being given with the rank that it occupies, there will be only to substitute into this formula for p the rank of the figurate number that one seeks, there is that which I see demonstrated quite at length & very scholarly in a posthumous Book of Mr. the Marquis de l'Hopital which appears some time ago, he employs this Theorem to establish many curious & new propositions in Geometry; but the way that he takes in order to demonstrate this Lemma is totally different from this here.

SECOND METHOD
FOR COMBINATIONS.

PROPOSITION XII.

In order to make myself more easily understood I serve myself with an example. Let be supposed that one seeks how much the odds be that drawing four cards at random from forty, for example in the game of Hombre, one will draw the four aces. It is first evident that it is permitted to me to suppose that these four aces will be found in the four cards above, since I have the liberty to choose them everywhere where I would wish; now it is clear that naming m the number of all the possible arrangements of forty cards, my lot in order that the ace of diamonds is found in the first place is $\frac{1 \times \frac{m}{40}}{m} = \frac{1}{40}$, since this ace being in the first place the thirty-nine others are able to have all the diverse arrangements imaginable; & likewise that my lot for that ace of diamonds being found in the first place, the ace of hearts is in the second, is $\frac{1 \times \frac{m}{40} \times \frac{1}{39}}{m} = \frac{1}{40 \times 39}$; since the ace of diamonds being in the first place, & the ace of hearts in the second, the thirty-eight other cards are able to be ranked diversely in as many ways as a number composed of 38 products of the natural numbers from unity to the number 38 inclusively expressed in units.

And for the same reason my lot in order to bring forth the ace of clubs in the third place, the ace of diamonds being in the first & the ace of hearts in the second, will be $\frac{1 \times \frac{m}{40} \times \frac{1}{39} \times \frac{1}{38}}{m} = \frac{1}{40 \times 39 \times 38}$: & finally my lot in order that, the ace of diamonds being in the first place, the ace of hearts in the second place, & the ace of clubs in the third, the ace of spades was in the fourth, is $\frac{1 \times \frac{m}{40} \times \frac{1}{39} \times \frac{1}{38} \times \frac{1}{37}}{m} = \frac{1}{40 \times 39 \times 38 \times 37}$. It is equally certain by the rule of the changes of order, page 5 that the product of the four numbers 1, 2, 3, $4 = 24$ expresses all the possible arrangements of four aces in the four first places. Therefore my lot in order that the four aces are found in the first four cards will be $\frac{1 \times 2 \times 3 \times 4}{40 \times 39 \times 38 \times 37}$. Now if I suppose that the letter p expresses the number of all the possible ways to take four things in forty, it is evident that my lot in order to take four determined things in forty, will be $\frac{1}{p}$: I will have therefore $\frac{1 \times 2 \times 3 \times 4}{40 \times 39 \times 38 \times 37} = \frac{1}{p}$, whence when I will draw $p = \frac{40 \times 39 \times 38 \times 37}{1 \times 2 \times 3 \times 4}$, & $\frac{1}{p} = \frac{1}{91390}$.

This example shows that if I myself propose to draw any number expressed by q of determined things, in a greater number called p, my lot will be expressed by a fraction of which the numerator will be composed of as many products of the natural numbers 1, 2, 3, 4, &c. & the denominator of as many products of the quantities p, $p-1$, $p-2$, $p-3$, &c. as

q expresses units; so that naming g the number of all the possible ways to take q in p, my lot will be $\frac{p}{g} = \frac{1\times2\times3\times4\times5\times6, \&c.}{p\times p-1\times p-2\times p-3\times p-4\times p-5, ,}$ whence I draw $g = \frac{p\times p-1\times p-2\times p-3\times p-4\times p-5}{1\times2\times3\times4\times5\times6}$; &c. *that which it was necessary to find*. And consequently here is fallen back to me by the method of changes of order in that of the combinations, & in the same formula of figurate numbers that we have found before by taking a quite different route.

PROPOSITION XIII.
LEMMA.

Pierre holding in his hands any number of tokens of all colors, white, black, red, &c. wagers against Paul that drawing at random any determined number of tokens, he will draw so many white, so many black, so many red, &c. One demands what is the lot of Pierre & the one of Paul in all the possible cases.

It is necessary to multiply the number which expresses in how many ways the white tokens that Pierre must take at random, are able to be taken differently in the number of white tokens proposed, by the number which expresses in how many ways the black tokens that Pierre must draw at random, are able to be taken differently in the entire number of black tokens proposed; to multiply next this product by the number which expresses in how many different ways the red tokens that Pierre has proposed to draw, are able to be taken in the red tickets proposed, to multiply anew this product, &c. & to divide all these products by the number which expresses in how many different ways all the tickets together of different color that one must take, are able to be taken in all the proposed tokens. The exponent of this division will express the lot of Pierre, or that which Pierre should wager against Paul in order that the division be equal.

In order to render the demonstration more facile & less abstract, I am going to apply it to some examples.

EXAMPLE I.

Pierre holds two white tokens, two black tokens & two red tokens. He wagers that having mixed them, & drawing next three tokens at random among these six he will draw from them one white, one black & one red.

It is necessary to remark that if there was in the game only two black tokens & two white tokens, he would have four different ways to take in these four tokens two tokens of different color; because each of the two white tokens would be able to be taken with each of the two black tokens, that which makes four ways. Now if to these four tokens one adds two other reds, if is clear that the four different ways that one just found being able to be encountered each with each of the two red tokens, the product which expresses all the coups that Pierre has in order to draw three tokens of different colors among these six, will be the cube of two, that is, that it will be necessary to multiply the number which expresses in how many ways a white token is able to be taken differently in two white tokens by the number which expresses in how many different ways a black token is able to be taken in two black tokens, & to multiply this product by the number which expresses in how many ways a red token is able to be taken differently in two red tokens, *that which it was necessary to demonstrate*. Now one will find by the twelfth Proposition & by the table, that six things are able to be taken differently three by three in twenty ways; & consequently the lot of Pierre will be expressed by the fraction $\frac{8}{20}$ or $\frac{2}{5}$, the lot of Paul will be therefore expressed by the fraction $\frac{3}{5}$, & consequently the lot of Pierre would be to the lot of Paul as two is to three, *that which it was necessary to find*.

EXAMPLE II.

Pierre holds fifty-two tokens in his hands, namely thirteen white, thirteen black, thirteen red & thirteen blue, or, that which reverts to the same, an entire deck composed of fifty-two cards. One demands in how many different ways he is able, drawing four cards at random in these fifty-two to draw one diamond, one heart, one spade & one club from them.

If there were only thirteen diamonds & thirteen hearts, there would be one hundred sixty-nine different ways to take in these twenty-six cards two cards of these two kinds; because each of these thirteen diamonds would be able to be taken with the ace of hearts, that which makes thirteen, or with the deuce of hearts, that which makes thirteen again, & thus consecutively each of the thirteen diamonds would be able to be taken with each of the thirteen hearts, that which makes 13×13, that is one hundred sixty-nine ways to take a diamond & a heart in twenty-six cards.

Presently if in these thirteen diamonds & in these thirteen hearts one adds thirteen clubs, it would be necessary to have all the possible ways to take a diamond, a heart & a club in these thirty-nine cards, to multiply by 13 the one hundred sixty-nine preceding ways; because each of these one hundred sixty-nine different ways will be able to be found with the ace of clubs, that which makes $13 \times 13 \times 1$, & with the deuce, that which makes $13 \times 13 \times 2$, that is three hundred thirty-eight different ways, & with three, that which makes five hundred seven different ways, & thus successively each of the one hundred sixty-nine preceding ways will be able to be found with each of the thirteen clubs, that which makes $13 \times 13 \times 13$, that is two thousand one hundred ninety-seven different ways to take a diamond, a heart & a club in these thirty-nine cards. One will observe likewise that the fourth power of thirteen will express in how many different ways four cards of different kinds, namely a diamond, a heart, a spade & a club are able to be taken in the fifty-two cards; *that which it was necessary to demonstrate*.

Now one will find by the preceding Problem & by the table that fifty-two cards are able to be taken four by four in two hundred seventy thousand seven hundred twenty-five ways; & consequently the lot of Pierre will be expressed by this fraction $\frac{13^4}{270725} = \frac{28561}{270725}$, & the lot of Paul by this other $\frac{242164}{270725}$, & consequently the lot of Pierre will be to the lot of Paul :: $28561 : 242164$ very near :: $1 : 8$, so that he will have the advantage to wager one against nine, & disadvantage to wager one against eight.

DEFINITION.

I will call single cards the cards of different kinds; double card, two cards of like kind, for example, two Kings, two Queens, two Jacks, &c. triple card, three cards of one same kind, for example, three aces, three Jacks, three tens, &c. quadruple card, four cards of one same kind, quintuple card, five cards of one same kind, &c.

PROBLEM
PROPOSITION XIV.

Let any number of cards be composed of an equal number of aces, of twos, of threes, of fours, &c. Pierre wagers that drawing at random among these cards a certain number of cards at will, he will draw from them so many singles, so many doubles, so many triples, so many quadruples, so many quintuples, &c.

In order to make understood more easily the solution of this Problem, I will call m the number of cards in which one wishes to take a certain number of them; q the number of times that each kind of cards is repeated in m; p the number of different kinds of cards, so that $q \times p$ is $= m$; b the exponent of the card which has the high dimension among those that one is proposed to take, c, d, e, f, &c. the exponent of the other cards that one wishes to take, of which the dimension is less, so that c expresses a number smaller than b, & d a number smaller than c, & e a number smaller than d, &c.

I will name also B the number of cards that one demands of the dimension expressed by b, C the number of cards that one demands of the dimension expressed by c, D the number of cards that one demands of the dimension expressed by d, &c.

I will express also by this mark $\overset{q}{\underset{b}{\square}}$ the number which expresses in how many ways b is able to be taken in q, setting the smallest number below, & the greatest above, & between two this arbitary mark $\square$. All this put, here is the method.

It is necessary to seek in how many ways b is able to be taken in q, raise this number to the exponent B, to multiply this product by the number which expresses in how many ways B is able to be taken in p. (2) To multiply this product by the number which expresses in how many ways, c is able to be taken in q, to raise this number to the exponent C, & to multiply this product by the number which expresses in how many way C is able to be taken in $p - B$. (3) To multiply the preceding products by the number which expresses in how many ways d is able to be taken in q, to raise this number to the exponent D, & to multiply it by the one which expresses in how many ways D is able to be taken in $p - B - C$. (4) To multiply all the preceding products by the number which expresses in how many ways e is able to be taken in q, to raise this number to the exponent E, & to multiply by the one which expresses in how many ways E is able to be taken in $p - B - C - D$, & thus consecutively. Finally it is necessary to divide this quantity formed of all these products by the number which expresses in how many ways one is able to take the number of cards proposed in m. The method is contained in this formula.

$$S = \overset{q}{\underset{b}{\square}}^{B} \times \overset{p}{\underset{B}{\square}} \times \overset{q}{\underset{c}{\square}}^{C} \times \overset{p-B}{\underset{C}{\square}} \times \overset{q}{\underset{d}{\square}}^{D} \times \overset{p-B-C}{\underset{D}{\square}} \times \overset{q}{\underset{e}{\square}}^{E} \times \overset{p-B-C-D}{\underset{E}{\square}} \times \overset{q}{\underset{f}{\square}}^{F} \times \overset{p-B-C-D-E}{\underset{F}{\square}} \times \&\text{c.}$$

which must be divided by the number which expresses in how many ways one is able to take the number of cards proposed in m.

It is necessary to remark that this formula being applied to a particular case, must be composed only of as many products as are here marked by $\times$, as there are cards of different dimensions among them as one wishes to take.

One would not be able to give the demonstration of this formula without making a long discourse, & one will not have much more difficulty to find it as one would have to understand it, thus wishing to leave it to the pleasure of the Reader, I will content myself to clarify it by some examples.

Example I. If Pierre is proposed to draw seven cards in fifty-two, so that there are three doubles & one single, one will have by the formula

$$\frac{\overset{4}{\underset{2}{\square}}^{3} \times \overset{13}{\underset{3}{\square}} \times \overset{4}{\underset{1}{\square}}^{1} \times \overset{10}{\underset{1}{\square}}}{4 \times 13 \times 17 \times 10 \times 7 \times 47 \times 46} = \frac{6^3 \times 13 \times 2 \times 11 \times 4 \times 10}{4 \times 13 \times 17 \times 10 \times 7 \times 47 \times 46} = \frac{16632}{900473}.$$

Example II. If Pierre was proposed to draw eight cards in fifty-two, so that there was a triple, two doubles & one single, one would have

$$S = \overset{4}{\underset{3}{\square}}^{1} \times \overset{13}{\underset{2}{\square}} \times \overset{4}{\underset{2}{\square}}^{2} \times \overset{12}{\underset{2}{\square}} \times \overset{4}{\underset{1}{\square}}^{1} \times \overset{10}{\underset{1}{\square}}$$

divided by $13 \times 25 \times 17 \times 7 \times 47 \times 46 \times 9 = \frac{266112}{40521285}$.

Example III. If Pierre was proposed to draw thirteen cards in two entire decks composed of 104 cards, so that there were two quadruples, two doubles & one single, one would have by the formula

$$
\begin{aligned}
S &= \frac{{\underset{4}{\overset{8}{\square}}}^{2} \times \underset{2}{\overset{13}{\square}} \times {\underset{2}{\overset{8}{\square}}}^{2} \times \underset{2}{\overset{11}{\square}} \times {\underset{1}{\overset{8}{\square}}}^{1} \times \underset{1}{\overset{9}{\square}}}{103 \times 34 \times 101 \times 14 \times 97 \times 95 \times 94 \times 31 \times 92} \\
&= \frac{70^2 \times 13 \times 6 \times 28^2 \times 11 \times 5 \times 72}{103 \times 34 \times 101 \times 14 \times 97 \times 95 \times 94 \times 31 \times 92}
\end{aligned}
$$

PROBLEMS

ON THE GAME OF PIQUET.

PROPOSITION XV.

Pierre is last in Piquet, & is supposed to have no ace at all. One demands what is his expectation in drawing either one, or two, or three.

One knows that in this game the Players each take twelve cards, that there remain eight of them in the stock, of which the first takes five, & the last three.

This supposed, one will find by propositions 11, 12, 13, that the lot of Pierre in order to to draw one ace in the three cards is $\frac{8}{19}$

That his lot in order to take two is $\frac{24}{285}$

That his lot in order to take three is $\frac{1}{285}$

And consequently, that his lot in order to take either one, or two, or three indeterminately is $\frac{29}{57}$

so that he is able to wager end-to-end with advantage that there will enter some of them, since the just division would make 29 against 28.

If one supposes that Paul who is first in card has no King at all, one will find

That his lot in order to have it is $\frac{455}{969}$

That his lot in order to have two is $\frac{70}{323}$

That his lot in order to have three is $\frac{10}{323}$

That his lot in order to have four is $\frac{1}{969}$

Therefore his lot in order to have any one indeterminately will be $\frac{292}{323}$; & consequently there is odds two hundred thirty-two against ninety-one, around five against two, that the first having no King at all, there will enter some one of them to him in five cards.

PROBLEM

PROPOSITION XVI.

Pierre is last, & is supposed to bear no diamond at all. One demands how much is the odds that there will return to him in his three cards what to prevent that Paul who is first is not able to have quinte or higher.

One will find by propositions 11, 12, 13, that there are two hundred twenty different coups which give the eighth to Paul:

That there are 132 which give to him a seventh, 168 which give to him a sixth, And finally 208 which give to him a quinte; And consequently the just part of the wager would be 103 against 182, that which would give a little less than three against five.

If one supposed that Pierre was first in cards, the other circumstances of the Problem remaining the same, one would find that there would be odds 10433 against 5071, that there would return to Pierre in the five cards which prevent that Paul could not have quinte, or sixth, or seventh, or eighth.

Because under this second supposition there will be 792 coups which will give an eighth to Paul,

990 which will give to him a seventh,

1650 which will give to him a sixth,

1639 which will give to him a quinte.

This Problem & the preceding would be able to be useful to the Players on some occasions, & to serve to determine them either in the manner to separate, or to propose or to accept with reason certain division, for example, to shuffle the cards, to give some points or the hand. They will be able also to serve as model in order to resolve an infinity of similar, which would be at least curious, if they are not each useful.

PROBLEM
ON THE GAME OF TRUMPS
PROPOSITION XVII.

Pierre & Paul play to five points in Trumps, they have each three of them, Pierre is first, he has the King & the third Queen of trumps, which will be for example clubs, & a King of diamonds guarded with the Jack: when he plays his King of trumps for the first card, Paul offers to him one point. One demands if he must accept it, & what is, in refusing it, his expectation to take all the tricks.

It is necessary first to examine in how many different ways it is able to happen that Paul has the Queen guarded by one or by many diamonds indeterminately, to subtract from this number the one which expresses in how many ways it is able to happen that Paul has the third Queen in diamonds, with another Queen guarded by some other color, & to subtract from it again the half of the number which expresses in how many ways it is able to arrive that Paul has the Queen guarded by diamond, another Queen guarded & any other fifth card of another kind. The number which will remain, these subtractions being made, will be the one which expresses how many coups there are which are able to prevent Pierre making all the tricks.

One will find by propositions 11, 12, 13, 14, that there are 3605 for the first case, 72 for the second, 240 for the third.

One will find also that the number which expresses in how many different ways one is able to take five cards in twenty-two, is 26334, & consequently one will have the lot of Pierre in this fraction $\frac{23041}{26334}$.

Thus the advantage of Pierre resulting from it the proposition of Paul will be expressed by this fraction $\frac{4937}{13167}A$. Therefore by supposing that A, which expresses the money of the game was two pistoles, if some one wished to buy the rights of Pierre & sit in his place, he must give to Pierre seven livres nine sols & eleven deniers beyond his stake.

It is easy to see thence that it is more advantageous to Pierre to attempt the slam, than to accept a point; because by accepting it his lot would be only $\frac{3}{4}A$, & even a little less, since there is appearance that in this game the primacy gives some advantage to a Player who has three points of five against the other four. Now it is evident that $\frac{3}{4}A$ is less than $\frac{23041}{26334}A$. Therefore, &c. This solution is able to be applied in some similar cases in the game of Hombre, & principally in Hombre with two.

PROBLEM
ON THE GAME OF HOMBRE
PROPOSITION XVIII.

Pierre makes to play in black, & is supposed to have any number of trumps. One demands what expectation he has to draw a certain number of trumps in the cards which he takes in the stock.

First Case.
Pierre has three trumps & takes six cards.

The expectation that he has to draw a trump at least in six cards, is expressed by the fraction $\frac{30254}{35061}$; thus he would be able to wager 30254 against 4807, that which is a little more than six against one.

The expectation that he has to draw from them at least two is expressed by the fraction $\frac{366142}{736281}$, so that there would be odds 366142 against 370139.

Second Case.
Pierre has four trumps, & takes five cards.

The expectation that Pierre has to draw at least one trump in these five cards is expressed by the fraction $\frac{18321}{24273}$; thus he would be able to wager 18321 against 5952, & he would have the advantage to wager three against one.

The expectation that he has to draw at least two trumps will be expressed by the fraction $\frac{53025}{169911}$, thus he would be able to wager 17675 against 38962.

Third Case.
Pierre has five trumps, & takes four cards.

The expectation that Pierre has to draw at least one trump in four cards is expressed by the fraction $\frac{4123}{6293}$, thus he would be able to wager 4123 against 2170, a little less than two against one.

It will be easy to resolve by propositions 11, 12, 13, a great number of other Problems of same kind as the one here, which would be able to serve to fix some rules for knowing in what game it is proper to play or to pass, or to play without taking. I would suffice for that to seek for the red cards that which one just found for the black, & to make the Kings, the differerent Matadors & the renounced enter into the calculation. One could able also determine in what game it is permitted to demand gano; but the extent of these matters obliges us to give ourselves some limits. It suffices to mark the path; thus I will finish with the following Problem which is rather easy, & will be able to be of some use.

PROBLEM II.
PROPOSITION XIX.

Pierre is first in cards, he has three Matadors in black, & any five other trumps. One demands how much it is necessary that he has in the game in order that there be more advantage to him to take in the stock than to play without taking.

I suppose that each Player gives a counter for the without drawing to the one who wins it.

It is necessary to remark that Pierre has three coups out of thirty-one in order to draw a trump, & three coups out of thirty-one in order to draw a King which is not trump, & that in each case the slam is assured to him. This supposed, if one names f each counter, p that which Pierre would win in playing without taking, & b that which would come to Pierre precisely from the slam.

It would be necessary in order that Pierre has reason to play without taking that $6 \times \overline{p-2f+b} + 25 \times \overline{p-2f}$ is greater than $31p$, & if one wishes to know in what case it would be indifferent to him to take or to play without taking. There is only to form this equality $6 \times \overline{p-2f+b} + 25 \times \overline{p-2f} = 31p$, & to deduce according to the ordinary rules $b = \frac{31}{3}f = 10f + \frac{1}{3}f$. Whence it is necessary to conclude that the profit of the slam must be at least of ten counters & six tokens in order that Pierre is able to take without disadvantage; & consequently if there is no joker at all in the game, Pierre will take his division by taking or not taking according as that which will be before each player will be either greater or lesser than 14 tokens, in supposing that one gives two counters for the slam.

If one would play the increase of the Matadors, & if the trumps of Pierre was three Matadors, the Queen, the Jack, the seven, the six & the five, the equation would be $5 \times \overline{p-2f+b} + 1 \times \overline{p+10f+b} + 25 \times \overline{p-2f} = 31p$, from which one would draw $b = 8f + \frac{1}{3}f$, that is, that it would be necessary for playing without taking that the profit of the slam was greater than eight counters & six tokens, & consequently if, not having at all the joker in the game, there is found more than nine tokens before each player, Pierre will have reason to take, & he will play without taking if he has nine or less than nine.

Let us suppose now that Pierre has four Matadors sevenths in spades, & two wrong ones which will be for example, the three of clubs & the five of diamonds.

In order to find how much he must have in the tricks under this supposition so that it is more advantageous to Pierre to take for the slam, than to play without taking.

I remark first that there are 24 coups which assure the slam to Pierre; because there are six in order to take two trumps, twelve in order to take a trump & a King, three in order to take two Kings, & three in order to take a King guarded by a Queen.

I observe next that there are 117 coups which render the lot of Pierre uncertain for the slam, namely, when there enters a guarded King to him, or a trump with a wrong one. I will call in this case his expectation x.

One will have $24 \times \overline{p-2f+b} + 117x + 324 \times \overline{p-2f} = 465p$

$$\text{or } b = \frac{465p - 348p + 696f - 117x}{24}$$

$$\text{or } b = \frac{969f + 117p - 117x}{24}$$

If one supposes $x = p - 2f + \frac{1}{5}b$, one will have $b = \frac{4650}{237}f = 19f + \frac{147}{237}f$. One will find thus different values of b according to all the different suppositions that one will make on the value of x. That which we just made seems to approach rather the truth. One would be able to find it exactly, but this would be a new Problem which would draw us too far. This Problem is easier, & of a more frequent use in regard to Hombre with two.

PROBLEM
ON BRELAN
PROPOSITION XX.

Pierre, Paul & Jacques play at Brelan, Pierre & Paul keep the deck, & Jacques passes. The card which returns is the King of hearts, Pierre is first, he has the ace & the King of diamonds, & the ace of hearts. Paul has the ace, the nine & the eight of clubs. Two of the Spectators, who have seen each of the games of Pierre & of Paul, & have not seen at all the one of Jacques, dispute in order to know which of the two Players Pierre & Paul has the better game, & the more expectation to win. One of the two, named Jean, wagers for

Pierre: the other, named Thomas, wagers for Paul. The money of the wager is called A. One demands what is the lot of the two Spectators Jean & Thomas, & that which they must each set into the game in order to wager without advantage or disadvantage.

It is necessary to note, (1) that Jean will win, if the three cards of Jacques are either three hearts or three diamonds. (2) That he will win again if one of the three was a spade or a club, the two others are either two hearts or two diamonds. (3) That if one of the three cards of Jacques is a heart or a diamond, the two others being clubs, Jean will have won. (4) That he will win again if the three cards of Jacques are a diamond, a heart & a spade, & that in each other disposition of the cards of Jacques he has lost.

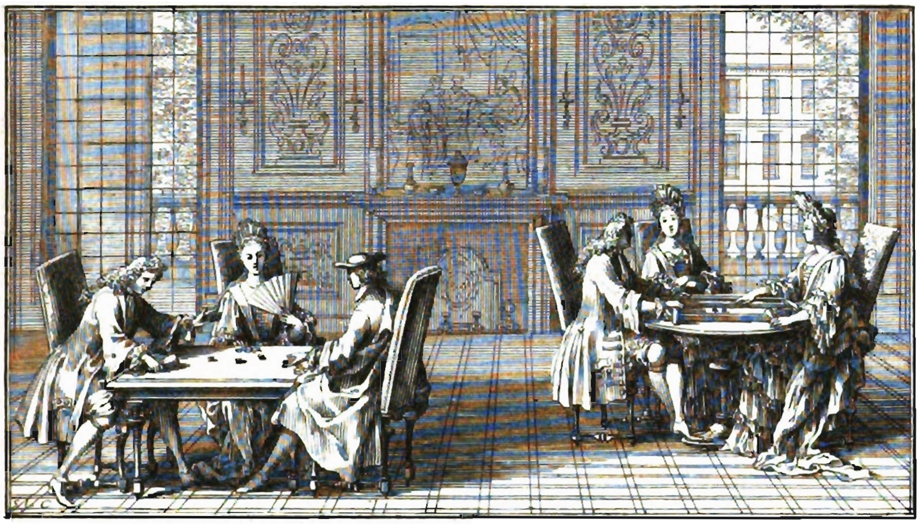
This supposed, there remains nothing more than to examine how many different chances there are, which give each of these four different dispositions of three cards of Jacques. One will find by propositions 11, 12, 13, 14, that there are twenty for the first, two hundred twenty for the second, two hundred ten for the third, & one hundred seventy-five for the fourth, & consequently the lot of Jean will be $\frac{125}{266}A = \frac{1}{2}A - \frac{4}{133}A$, that which shows that the condition of Pierre is less advantageous than that of Paul, & that Jean in order to wager equally against Thomas, must set into the game 125 against 141.

PROBLEM
ON IMPERIAL
PROPOSITION XXI.

In order to have an Imperial in the game which bears this name, it is necessary to have either four aces, or four Kings, or four Queens, or four Jacks, or four sevens, or major fourth, or carte blanche. One demands how much a Player is able to wager that there will come to him a determined Imperial, for example an Imperial of ace, or carte blanche.

One will know by propositions 12, 13, 14, that out of the number 225792840, which expresses in how many ways one is able to take twelve cards out of thirty-two, there are 3108105 of them in order to have an Imperial of aces, & 125970 in order to have carte blanche.

The lot of a Player, who would wager in Imperial or in Piquet in order to have carte blanche, would be therefore expressed by the fraction $\frac{323}{578956}$, thus he would have the advantage to wager 1 against 1792, & disadvantage to wager 1 against 1791.



PROBLEM
ON
QUINQUENOVE

SECOND PART
EXPLICATION OF THIS GAME.

One draws first among the Players for whom will have the dice box. Let us suppose it falls to Pierre; & in order to make the Game understood more easily, let us suppose there are only two Players, Pierre & Paul. The latter will set first a certain sum into the game; then Pierre casting the dice, here is that which happens. If Pierre brings forth five or nine, he loses, & gives the dice box to Paul. If Pierre brings forth either three, or eleven or a doublet, he draws the stake of Paul. The latter remits into the game, & Pierre continues to play. If Pierre brings forth any of the preceding coups, he will neither lose, nor win. In order to explicate that which happens in this case, let us suppose, for example, that Pierre has brought forth seven on the first coup. One will note, (1) that Pierre replaying, will be able to win the stake from Paul only by bringing forth seven. (2) That Paul has the liberty to risk a new stake, & that Pierre will be similarly at liberty to take it, or to not take it. (3) That Paul in order to distinguish this stake from the preceding, set it below, & that it is named masse.(4) That if this masse is equal to the stake, it is named masse in the game; & that when it is not the same, it is named masse to the dice. (5) That Pierre having accepted this new masse, he will win by bringing forth the following coup, either three, or eleven, or doublet, or else by bringing forth in sequence this chance before that of bringing forth five or nine; but that he is able to win the first stake which is said entered into the game, only by bringing forth seven; & finally that he will lose them all both by bringing forth either five or nine.

Let us suppose presently for a more ample explication, that Pierre having said, *Taupe à la masse*, brings forth on his second coup eight otherwise than by doublet, that is to say by six & two, or by five & three, & that Paul sets into the game a new masse that Pierre accepts. One will note, (1), that Pierre will win this masse by bringing forth either three, or eleven, or doublet. (2) That he will win the first stake of Paul by bringing forth seven, &

the second by bringing forth eight. (3) That he loses the two stakes & the masse in bringing forth either five or nine, & that then he cedes the dice box to Paul.

That which I just explicated for a small number of coups, & only in regard to two Players, must be understood of every other number of coups & of Players.

PROBLEM
PROPOSITION XXII.

Pierre & Paul play at Quinquenove, & Pierre holds the dice box. I suppose that the stake of Paul is always the same, & expressed by A. I suppose also that Pierre will not accept masse at all, but that he will be obliged to hold the game until he has lost; after which I suppose the game ended. One demands what is in this game the advantage & the disadvantage of the one who has the die; or, that which reverts to the same, how much Pierre should demand or give to a third in order to cede to him the dice box, & give to him to play in his place.

SOLUTION.

The lot of Pierre when he cast the die, is to have eight coups in order to lose, namely five which arrives in four ways, & nine which arrives similarly in four ways; to have ten coups in order to win, namely the six doublets, three, which arrives in two ways, & eleven which arrives similarly in two ways; to have four coups in order to bring forth six otherwise than by doublet, so many to bring forth eight otherwise than by doublet, two coups in order to bring forth four otherwise than by doublet, two coups in order to bring forth ten otherwise than by doublet, & finally six coups in order to bring forth seven.

Therefore if I name x the lot of Pierre when he has brought forth eight or six, z his lot when he has brought forth four or ten, y his lot when he has brought forth seven, q the advantage or the disadvantage that Pierre finds to continue the game when he has won, & S his lot in general. One will have the sought lot of Pierre

$$S = \frac{10 \times \overline{2A+q} + 8x + 4z + 6y}{36}$$

It is necessary presently to seek the values of the unknowns x, z, y & q.

In order to determine the value of the unknown x, I note that Pierre having brought forth on the first coup six without doublet, he has by replaying five coups in order to win it, eight coups in order to lose, & twenty-three coups in order to replay.

One will have therefore $x = \frac{5}{13} \times \overline{2A+q}$.

One will find likewise $y = \frac{3}{7} \times \overline{2A+q}$. And $z = \frac{3}{11} \times \overline{2A+q}$.

Now if one substitutes these values of x, y, z in the equality proposed, one will have

$$\begin{aligned} S &= \frac{10 \times \overline{2A+q} + 8 \times \frac{5}{13} \times \overline{2A+q} + 4 \times \frac{3}{11} \times \overline{2A+q} + 6 \times \frac{3}{7} \times \overline{2A+q}}{36} \\ &= \frac{10 \times \overline{2A+q} + \frac{6746}{1001} \times \overline{2A+q}}{36} = \frac{4189}{9009} \times 2A + \frac{4189}{9009} q \end{aligned}$$

In order to know the value of q, it is necessary to note that if q were $= 0$, that which would happen if Pierre & Paul agreed that the game must end as soon as Pierre would have won. Then the lot of Pierre would be $\frac{8378}{9009} A = A - \frac{631}{9009} A$; whence it is clear that the quantity $\frac{631}{9009} A$ would express the disadvantage that Pierre would have in this game.

One will observe similarly that if Pierre & Paul agreed before playing, that Pierre having won once, will continue to play until this that he has either won anew or lost, the disadvantage of Pierre would be $\frac{631}{9009} A + \frac{4189}{9009} \times \frac{631}{9009} A$.

And if one supposes indeterminately according to the rule of this game, that Pierre will continue to hold the Bank until this that he has lost, his disadvantage will be expressed by this infinite series $\frac{631}{9009}A + \frac{4189}{9009} \times \frac{631}{9009}A + \overline{\frac{4189}{9009}}^2 \times \frac{631}{9009}A + \overline{\frac{4189}{9009}}^3 \times \frac{631}{9009}A + \overline{\frac{4189}{9009}}^4 \times \frac{631}{9009}A+$&c. The sum of this series is $= \frac{1}{8}A + \frac{57}{9640}A = 1$ l. 6 f. $2\frac{46}{241}$ d. supposed that the stake of Paul was a pistole.

And this would be thence the disadvantage of Pierre if he played against a player who at each time that he lost set A into the game, & of which Pierre never holds any masse.

Thus Pierre is able to count that out of each pistole that one of the players sets into the game, whether it is a stake or a masse, there is for him 14 f. $\frac{74}{9009}$ of pure loss, that which is a little more than the fifteenth part of his stake, & a little less than a fourteenth. *That which it was necessary to find.*

This advantage is rather considerable, principally when there is a great number of Players, in order to gratify those who keep the die by refusing the masses, that which takes off all the agreement of this game. It would be therefore proper to reform it by rendering it more equal, & by giving a little advantage to the one who holds the die, in order to engage him to keep the masses. For that if would be necessary to agree that the number 4 brought forth at the second coup, won as well as 3 & 11. Then the advantage of the one who holds the die with respect to the stake of each player, would be expressed by the fraction $\frac{97}{9009}$, which is very nearly the ninety-third part of unity.

PROBLEM
ON THE GAME OF HAZARD
EXPLICATION OF THIS GAME

One plays with two dice as in Quinquenove. Let us name Pierre the one who holds the die, & let us suppose that Paul represents the other Players. Pierre will cast the die until this that he has brought forth either 5, or 6, or 7, or 8, or 9; the one of these numbers which will be presented first will serve as chance to Paul, next Pierre will recommence to cast the die in order to give his chance. Now the chances of Pierre are either 4, or 5, or 6, or 7, or 8, or 9, or 10, so that he has two more of them than Paul, namely 4 & 10. It is necessary next to know that which follows:

(1) If Pierre after having given to Paul a chance which is either 6 or 8, brought forth at the second coup either the same chance, or twelve, he wins; but if he brings forth either bezet, or two & ace, or eleven, he loses.

(2) If he has given to Paul the chance of 5 or of 9, & if he brings forth at the following coup the same chance, he wins; but if he brings forth either bezet, or two & ace, or eleven, or twelve, he loses.

(3) If he has given to Paul the chance of 7, & if he brings forth the following coup either the same chance, or eleven, he wins; but if he brings forth either bezet, or two & ace, or twelve, he loses.

(4) Pierre being given a different chance from that of Paul, he will win if he brings forth his chance before bringing forth that of Paul, & he will lose if he brings forth the chance of Paul before bringing forth his own.

(5) When Pierre & Paul have lost, one recommences the game, by giving chances anew; but Pierre quits the die in order to give it to the one who follows him, only when he has lost.

(6) If there are many Players, they have each the same chance.

PROBLEM
PROPOSITION XXIII.

One demands what is in this game the advantage or the disadvantage of the one who holds the die.

Let it be supposed that the stake of Paul is $\frac{1}{2}A$.

(1) If the chance of Paul is 6 or 8, the lot of Pierre will be

$$\frac{6A + 8 \times \frac{4}{9}A + 6 \times \frac{6}{11}A + 6 \times \frac{3}{8}A + \frac{5}{2}A}{36} = \frac{6961}{14256}A.$$

(2) If the chance of Paul is 7, the lot of Pierre will be

$$\frac{8A + 8 \times \frac{4}{10}A + 6 \times \frac{3}{9}A + 10 \times \frac{5}{11}A}{36} = \frac{244}{495}A.$$

(3) If the chance of Paul is 5 or 9, the lot of Pierre will be

$$\frac{4A + 4 \times \frac{1}{2}A + 10 \times \frac{5}{9}A + 6 \times \frac{3}{7}A + 6 \times \frac{6}{10}A}{36} = \frac{1396}{2835}A.$$

Consequently the lot of Pierre will be

$$\frac{10 \times \frac{6961}{14256}A + 6 \times \frac{244}{495}A + 8 \times \frac{1396}{2835}A}{24},$$

& his advantage will be

$$\frac{10 \times \frac{167}{14256}A + 6 \times \frac{7}{990}A + 8 \times \frac{43}{5670}A}{24} = \frac{37}{4032}A.$$

This fraction which expresses the disadvantage of Pierre with respect to the stake of Paul, is smaller than $\frac{1}{108}$, & greater than $\frac{1}{109}$.

But because this disadvantage continues as long as Pierre continues to have the die, the disadvantage of Pierre considered in general is expressed by an infinite series of which the sum is $\frac{37}{2053}A$, so that if $\frac{1}{2}A$ designates a pistole, there is 3 f. $8\frac{1}{21}$ d. of pure loss for him out of each pistole, & Pierre would be able without disadvantage to give 7 f. $2\frac{1042}{2053}$ d. to the one who would offer to keep the die in his place.

REMARK I.

It is the custom of Players in this game to put their money only when one has delivered chance to them. Now it is evident that this usage is prejudicial to the one who holds the die, because since his disadvantage is around $\frac{1}{85}$, when the chance of the Players is 6 or 8, & only $\frac{1}{131}$ when their chance is either 5 or 9, & $\frac{1}{141}$ when their chance is 7, it is clear that if the Players knew with exactitude their interest, they would hazard more money when their chance is either 5 or 9, than when it is 7, & more yet when it is 6 or 8, than when it is 7 or 5 or 9. It would be therefore proper that the Players set their money into the game before the one who holds the die had delivered chance to them.

REMARK II.

One sees that this game is equal enough, but it would be more if one agreed that Pierre having brought forth 7 on the first coup, won on the second coup by bringing forth either the same chance, or 11 or 12, & that he lost only by bringing forth either bezet or two & ace; for I find that by this reform the one who holds the die would have advantage, but this would only be of one sol & two deniers out of each pistole, that which is not very considerable.

PROBLEMS
ON THE GAME OF ESPERANCE
EXPLICATION OF THIS GAME

One plays with two dice. The Players agree to take a certain number of tokens, & draw next to whom will have the die. This done, if the one who has the die brings forth an ace, he gives a token to the one who is to his left; if he brings forth a six, he sets a token into the game; if he brings forth six & ace, & if he has more than one token, he will pay one of them to his left & one to the game: but if he has only one of them, he will set it into the game. In all these cases the one who has the die after having paid, cedes the dice box to the one which follows him to the right. If he brings forth a doublet, he has the liberty either to replay in the expectation to bring forth again two doublets consecutively, that which would make him win it, or to cede the die to the one who follows him to the right. If he brings forth any other coup, that is, if he brings forth neither ace, nor six, nor doublets, he cedes the dice box, without paying anything to the one who is to his right; finally the former wins the money of the game, who first brings forth three doublets in sequence, or who conserves some token, all the other Players having lost theirs.

There is to note that when one has no more tokens, one no longer plays, & one is able to reenter the game (that which is named resuscitating) only by the help of the one who one has for neighbor to the right when he brings forth an ace.

PROBLEM
PROPOSITION XXIV.

Pierre, Paul & Jacques each take one token, & agree that the one who will remain with some tokens, the others no longer having them, will win a certain sum on which they agree. One supposes that Pierre has the die, that Paul is to his right & Jacques to his left. One demands what is the lot of the three Players, that is, what is the advantage & the disadvantage that the situation & the place where he is found gives to each of them.

Let A be the money of the game, & S the lot of Pierre at the commencement of the game.

It is necessary to remark first that Pierre casting the die has six coups out of thirty-six in order to bring forth a doublet, (2) that he has two coups in order to bring forth six & ace; (3) eight coups in order to bring forth ace with one die, the other die being neither ace nor six; these eight coups are ace & two, ace & three, ace & four, ace & five, two & ace, three & ace, four & ace, five & ace; (4) eight coups in order to bring forth a six with one die, the other die being neither ace nor six; these eight coups are six & two, six & three, six & four, six & five; two & six, three & six, four & six, five & six.

This supposed, it is clear that the lot of Pierre when he is going to play is, (1) having eight coups, so that having nothing Jacques has two tokens, & Paul one token & the die, namely when Pierre brings forth an ace with neither six nor doublet; (2) having ten coups, so that having nothing, Jacques has one token & Paul one token & the die, namely when Pierre brings forth a six without doublet; (3) having eighteen coups, so that Pierre, Paul & Jacques, having each one token, Paul has the die.

If one names x the lot of Pierre in the first case, y his lot in the second case, z his lot in the third case, one will have $S = \frac{8x+10y+18z}{36} = \frac{4x+5y+9z}{18}$.

In order to determine x, it is necessary to note that Pierre having nothing, Jacques having two tokens, & Paul having one token & the die, the lot of Pierre when Paul is going to cast the die is, (1) having eight coups, so that Paul having nothing, Pierre has one token, & Jacques two tokens & the die, namely when Paul brings forth one ace with neither six nor doublet; (2) ten coups which make him lose & finish the game, namely when Paul brings forth a six without doublet; (3) having eighteen coups, so that Pierre having nothing, Jacques has two tokens & the die, & Paul one token.

If one names u the lot of Pierre in the first case, & t his lot in the third, one will have $x = \frac{8u+10\times 0+18t}{36} = \frac{4u+9t}{18}$.

In order to determine u, it is necessary to observe that Paul having nothing, Pierre having one token, & Jacques two tokens & the die, the lot of Pierre when Jacques is going to cast the die is, (1) having two coups, so that Jacques having nothing, Paul has one token, & Pierre one token & the die, namely when Jacques brings forth a sixth & ace; (2) eight coups which resets the game as in the beginning, namely when Jacques brings forth an ace with neither six nor doublet; (3) eight coups, so that Paul having nothing, Jacques has one token, & Pierre one token & the die, namely when Jacques brings forth a six with neither ace nor doublet; (4) eighteen coups so that Paul having nothing, Jacques has two tokens, & Pierre one token & the die.

If one names p the lot of Pierre in the first case, q his lot in the third case, r his lot in the fourth case, one will have $u = \frac{2\times p+8\times S+8\times q+18\times r}{36} = \frac{p+4S+4r+9r}{18}$.

In order to determine p, let l be the lot of Pierre when Jacques having nothing, Pierre has one token, & Paul one token & the die, one will have $p = \frac{10\times 0+8\times y+18l}{36}$. Now $l = \frac{18\times A+18\times p}{36}$. Therefore $p = \frac{4y+9\times\overline{\frac{A+p}{2}}}{18} = \frac{8y+9A+9p}{36}$; whence one draws $p = \frac{8y+9A}{27}$, & $l = \frac{1}{2}A + \frac{1}{2} \times \frac{8y+9A}{27} = \frac{18A+4y}{27}$.

In order to determine q, let the lot of Pierre be called K when Paul having nothing, Pierre has one token, & Jacques one token & the die, one will find $q = \frac{18\times 0+18\times K}{36}$, & $K = \frac{10\times A+8\times p+18\times q}{36}$; whence one draws $q = \frac{5A+4p}{27}$. Now one has had above $p = \frac{8y+9A}{27}$; therefore $q = \frac{5A+4\times\frac{8y+9A}{27}}{27} = \frac{171A+12y}{27\times 27}$.

It is evident that $r = \frac{18\times 0+18\times u}{36} = \frac{1}{2}u$.

One will have (by substituting these values of p, q, r)

$$u = \frac{2\times\frac{8y+9A}{27} + 8\times S + 8\times\frac{171A+32y}{27\times 27} + 18\times\frac{1}{2}u}{36},$$

that which reduces to $u = \frac{688y+1854A+5832S}{19683}$; therefore $r = \frac{344y+927A+2916S}{19683}$.

In order to determine the value of t, it is necessary to note that Pierre having nothing, Paul having one token, & Jacques two tokens & the die, the lot of Pierre when Jacques is going to cast the die is, (1) having two coups in order to lose, namely when Jacques brings forth six & ace; (2) eight coups, so that Pierre having nothing, Jacques has one token, & Paul two tokens & the die, namely when Jacques brings forth one ace with neither six nor doublet; (3) eight coups, so that Pierre having nothing, Jacques has one token, & Paul one token & the die, namely when Jacques brings forth a six with neither ace nor doublet; (4) eighteen coups, so that Pierre having nothing, Jacques has two tokens, & Paul one token & the die.

This put, if one names c the lot of Pierre in the second case, one will have $t = \frac{2\times 0+8\times c+8\times y+18\times x}{36} = \frac{4c+4y+9x}{18}$.

In order to determine the value of c, one will note that Pierre having nothing, Jacques having one token, & Paul two tokens & the die, the lot of Pierre when Paul is going to cast the die is, (1) having two coups, so that Paul having nothing, Pierre has one token, & Jacques one token & the die, namely when Paul brings forth six & ace; (2) having eight coups, so that the three Players have each one token, Jacques having the die, namely when Paul brings forth one ace with neither six nor doublet; (3) having eight coups, so that

Pierre having nothing, Paul has one token, & Jacques one token & the die, namely when Paul brings forth one six with neither ace nor doublet; (4) having eighteen coups, so that Pierre having nothing, Paul has two tokens, & Jacques one token & the die.

If one names the lot of Pierre in the second case m, in the third case n, in the fourth b, one will have

$$c = \frac{2 \times K + 8m + 8n + 18b}{36} = \frac{K + 4m + 4n + 9b}{18}.$$

In order to determine m, let h be the lot of Pierre when Jacques having nothing, Paul has two tokens, & Pierre one token & the die, one will have

$$m = \frac{10 \times p + 8 \times h + 18 \times S}{36} = \frac{5p + 4h + 9S}{18}.$$

In order to determine h, let B be the lot of Pierre when Jacques having nothing, Pierre has one token, & Paul two tokens & the die, one will have

$$h = \frac{10 \times 0 + 8 \times c + 18 \times B}{36} = \frac{4c + 9B}{18}.$$

In order to determine B, let D be the lot of Pierre when Jacques having nothing, Paul has one token, & Pierre two tokens & the die, one will have

$$B = \frac{2 \times A + 8 \times p + 8 \times D + 18h}{36} = \frac{A + 4p + 4D + 9h}{18}.$$

In order to determine D, let E be the lot of Pierre when Jacques having nothing, Pierre has two tokens, & Paul one token& the die, one will have

$$\begin{aligned} D &= \frac{2 \times y + 8 \times z + 8l + 18 \times E}{36} = \frac{y + 4z + 4l + 9E}{18}, \\ \& E &= \frac{18 \times A + 18D}{36} = \frac{A + D}{2}. \end{aligned}$$

If one substitutes the values of E, D, B, h, into the preceding equations, one will find

$$\begin{aligned} D &= \frac{86y + 387A + 216z}{27 \times 27}. \\ B &= \frac{2A + 8p + 8D + 4c}{27}. \\ h &= \frac{8c + A + 4p + D}{27}. \end{aligned}$$

And finally $m = \frac{151p + 243S + 32c + 4A + 16D}{27 \times 18}$.

One will find also

$$\begin{aligned} n &= \frac{18 \times 0 + 18 \times y}{36} = \frac{1}{2}y. \\ b &= \frac{18 \times 0 + 18 \times c}{36} = \frac{1}{2}c. \end{aligned}$$

In order to determine the value of y, it is necessary to note that Pierre having nothing, Jacques having one token, & Paul one token & the die, the lot of Pierre is, (1) having eight coups so that Paul having nothing, Pierre has one token, & Jacques one token & the die, namely when Paul brings forth one ace with neither six nor doublet; (2) ten coups in order to lose, namely when Paul brings forth one six; (3) eighteen coups, so that Pierre having nothing, Paul has one token, & Jacques one token & the die.

This put, one will have $y = \frac{8\times K+10\times 0+18n}{36} = \frac{4K+9n}{18}$; & substituting into this equation for K its value $\frac{5A+4p+9q}{18}$, & for n its value $\frac{1}{2}y$, one will have

$$y = \frac{4\times 5A + 4\times \frac{\frac{8y+9A}{27}+9\times\frac{171A+12y}{27\times 27}}{18} + \frac{9}{2}y}{18},$$

whence one draws, by transposing & reducing, $y = \frac{2736}{19171}A$.

In order to determine the value of z, I note that Pierre, Paul & Jacques having each one token, & Paul having the die, the lot of Pierre is, (1) having ten coups, in order than Paul having nothing, he has one token, & Jacques one token & the die, namely when Paul brings forth a six without doublet; (2) eight coups, so that Paul having nothing, he has two tokens, & Jacques one token & the die, namely when Paul brings forth one ace with neither six nor doublet; (3) eighteen coups, so that Pierre, Paul & Jacques having each one token, Jacques has the die.

If one names G the lot of Pierre in the second case, one will have $z = \frac{10\times K+8\times G+18\times m}{36} = \frac{5K+4G+9m}{18}$.

In order to determine G, I note that Paul having nothing, Pierre having two tokens, & Jacques one token & the die, the lot of Pierre is, (1) having ten coups in order to win, namely when Jacques brings forth one six without doublet; (2) eight coups, so that Jacques having nothing, Paul has one token, & Pierre two tokens & the die, namely when Jacques brings forth one ace, with neither six nor doublet; (3) eighteen coups, so that Paul having nothing, Jacques has one token, & Pierre two tokens & the die.

If one names F the lot of Pierre in the last case, one will have $G = \frac{10\times A+8\times D+18\times F}{36}$.

In order to determine F, let L be the lot of Pierre when Paul having nothing, Pierre has one token, & Jacques two tokens & the die, one will have

$$F = \frac{2\times 0+8\times L+8\times K+18\times G}{36} = \frac{4L+4K+9G}{18}.$$

One will find also

$$L = \frac{8\times S+2\times p+8\times Q+18\times r}{36} = \frac{4S+p+4q+9r}{18}.$$

If one substitutes the values of G, of m & K into the equality $z = \frac{5K+4G+9m}{18}$, one will have

$$z = \frac{153351A+160299p+115263q+182331S+6780y+11664F+23328c+17280z}{708598},$$

& substituting anew for r its value $\frac{344y+927A+2916S}{19683}$, & for c its value

$$\frac{134847A+519048p+714092y+177147q+708588S+13824z}{4257657},$$

one will find, by transposing & reducing,

$$z = \frac{11643201030807915088}{23780027602988823717}A.$$

One will have also

$$m = \frac{7117013810993376514}{23780017602988823717}A;$$

& consequently

$$S = \frac{5029812761187532115}{23780027602988823717}A.$$

Thus the lot of the three Players Pierre, Paul & Jacques, will be as the three numbers 5029812761187532115, 7117013810993376514, 11643201030807915088.

It is necessary to note that in this Problem one has not had place to examine if it was advantageous to the Player to recommence when he brings forth on his coup a doublet, in the expectation to bring forth three of them in sequence; but this consideration would take place if in supposing (thus as it is made often) that it suffices in order to win to bring forth two doublets consecutively, there was found a greater number of players or only two Players who had each many tokens. One could give below some certain rules, thus as one is going to see in the Problem which follows.

PROBLEM II.
PROPOSITION XXV.

Pierre & Paul have any number of tokens. One demands in what case they must recommence when they bring forth a doublet. One supposes that they will win in bringing forth two doublets in sequence.

FIRST CASE.

Pierre & Paul have only one token each, & it is to Pierre to play. One demands what is his lot.

In order to resolve this Problem, it is necessary to make some assumptions touching the manner of play of Pierre & of Paul, because it is able to happen, (1) that Pierre & Paul recommence when they will have a doublet; (2) that they recommence in this case neither the one nor the other; (3) that Pierre recommences, & that Paul does not recommence; (4) that Pierre not recommence, & that Paul recommences. Now according to all these different suppositions, the lot of Pierre will be different.

(1) If the design of Pierre & of Paul is to not recommence when they will have a doublet, the lot of Pierre will be $\frac{1}{3}A$, & the one of Paul $\frac{2}{3}A$.

(2) If the design of Pierre & of Paul is to recommence when they will have a doublet, the lot of Pierre is $\frac{3}{10}A$, & the one of Paul $\frac{7}{10}A$.

(3) If the design of Pierre is to not recommence, & the one of Paul to recommence, in case of doublet his lot will be $\frac{21}{58}A$, & the one of Paul $\frac{37}{58}A$.

(4) If the design of Pierre is to recommence, & the one of Paul to not recommence, his lot will be $\frac{8}{29}A$, & the one of Paul $\frac{21}{29}A$.

It follows thence that Pierre, & consequently Paul, must cede the dice box without recommencing when they have brought forth a doublet.

In order to be assured if Pierre & Paul must recommence when they have brought forth a doublet, it suffices to examine if the lot of Pierre is greater or lesser when they both recommenc, than when neither one nor the other recommence.

SECOND CASE.

Pierre has one token against Paul two tokens, & it is to Pierre to play.

(1) If one supposes that neither Pierre nor Paul recommence when they will have one token against two, & that they will have brought forth a doublet, one will find that the lot of Pierre is $\frac{19}{105}A$, & the one of Paul $\frac{86}{105}A$.

(2) If one supposes that they both recommence when having one token against two, they will have brought forth a doublet, the lot of Pierre will be $\frac{1162}{6993}A$, & the one of Paul $\frac{5831}{6993}A$.

THIRD CASE.

Pierre & Paul have each two tokens, & Pierre has the die.

(1) If one supposes that neither the one nor the other will recommence when they will have one token against three, one will find the lot of Pierre $= \frac{30763}{73185}A$, & the one of Paul $\frac{42422}{73185}A$. One will find also that the lot of Pierre when he has one token against three, & that it is to him to play, is $\frac{16498}{73185}A$.

(2) If one supposes that both recommence when holding one token against three, they will have brought forth a doublet, the lot of Pierre will be $\frac{14989417}{35436135}A$, & the one of Paul $\frac{20446718}{15436135}A$.

It follows thence that Pierre must not recommence, when having one token against Paul three tokens, he brings forth a doublet.

One would be able in this way to examine if Pierre must either cede the die to Paul, or recommence when he has one token against four, or two against three, the calculation will be the same as the one of this Problem & of the preceding, but the length of it would be excessive, thus I advise no person to attempt it. There is much appearance that Pierre must recommence & to try to win by bringing forth two doublets consecutively, when having one token against four, he has brought forth a doublet; because I find that in the third & last case, the difference of the lot of Pierre when he not recommence, to his lot when he recommences, is $\frac{234488932}{24698986095}A$, that which is less than one hundredth.

PROBLEMS
ON TRICTRAC.

It is very useful, in order to play Trictrac agreeably & with advantage, to know at each coup of the die, the expectation that one has either of strike, or to replenish, or to cover some one of his dames [pieces] by the coup which one is going to play. This is also that which enough of the good Players know; but it is only by a great application & much exercise that one is able to acquire the custom for the cases which are a little composite. For example, there are few persons who are able to see at a glance that their small jan [table] being disposed thus on the side A of Trictrac, they have one coup in order to win twelve points, ten coups in order to win eight of them, three coups in order to win six, sixteen coups in order to win four of them, & finally six coups in order to not replenish. But that which passes extremely the ordinary knowledge of the Players & that which would be to them nevertheless very important in order to play well the dames, & to make some opportune holds, it is to be able to know with exactitude the expectation that one has to hold a certain number of coups without breaking, or to arrange his game in such or such fashion, in two or many coups. One is able to discover all these things by the preceding methods: Here are two quite simple Examples of it, of which the last is able to have some utility.

PROBLEM I.
PROPOSITION XXVI.

Pierre wagers that he will take his great corner in two coups. One demands that which he must win in order that the division be equal.

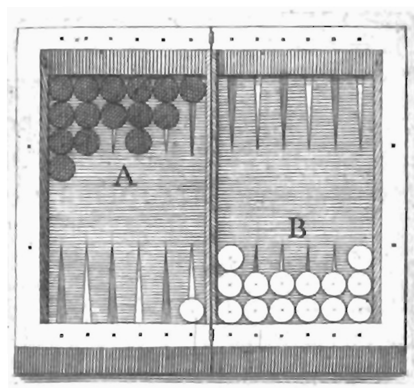
It is necessary to note, (1) that Pierre is able to win only by bringing forth from the first coup of die one of these four coups, six five, quine or sonnez.

(2) That having brought forth one of these four coups, he has not yet won; but that having brought forth six five on the first coup, he must in order to win bring forth again six five on the second coup; & that having brought forth on the first coup quine, he must in order to win bring forth on the second coup sonnez; & that having brought forth on the first coup sonnez, he must in order to win bring forth on the second coup either quine or sonnez. It follows from all this that the lot of Pierre will be $\frac{2}{36} \times \frac{2}{36} + \frac{1}{36} \times \frac{1}{36} + \frac{1}{36} \times \frac{2}{36} = \frac{7}{1296}$; thus

Pierre in order to wager without disadvantage, must set into the game 7 against 1289, & he would have the advantage to wager 1 against 186 to take his great corner in two coups.

PROBLEM II.
PROPOSITION XXVII.

My dames being disposed thus as it appeared in side B of Trictrac, I wish to know how much I should wager to hold two coups without breaking.



The chances of two coups are here mixed together, & ought be considered not independently at all from one another. As advantageous as my first coup is able to be , it is clear that my second coup is able to make me lose; & on the contrary as disadvantageous as it is, it removes from me the expectation to hold at the second coup. The greatest part of the coups of die that I am able to bring forth at the first coup, diversifies my attention for the event of the second, but there are of them which leave me an equal expectation. For example it is indifferent to me to bring forth at the first coup sonnez or five & ace, or four & two; six three or five & four, &c. In order to untangle all that, it is necessary to seek what is my expectation to hold at the second coup under all the different suppositions of the different coups of die that that I am able to bring forth at the first coup. The sum of all these chances will express my lot, one will find that I have, (1) two coups which give me $\frac{1}{36}$, namely six & five.

(2) Three coups which give me $\frac{3}{36}$, namely six four & quine, since having brought forth six four or quine on the first coup, I have in order to hold sonnez & six & ace.

(3) Four coups which give me $\frac{6}{36}$, namely six three, & five & four, because I will have in order to hold sonnez, six & ace, six two & bezet.

(4) Four coups which give me $\frac{10}{36}$, namely six two, & five & three, because I have in order to hold sonnez, six & ace, six two, six three, two & ace & bezet.

(5) Two coups which give me $\frac{12}{36}$, namely four & three, because I will have in order to hold at the second coup sonnez, six & ace, six two, six three, two & ace, three & ace & bezet.

(6) Four coups which give me $\frac{15}{36}$, namely six & ace, & five & two, because I will have in order to hold six & ace, sonnez, six two, bezet, six three, two & ace, six four, three & ace, & double two.

(7) Six coups which give me $\frac{21}{36}$, namely sonnez, five & ace, four & two & terns, because I will have in order to hold sonnez, six & ace, six two, bezet, six three, two & ace, six four, three & ace, double two, six five, four & ace,three & two.

(8) Four coups which give me $\frac{23}{36}$, namely four & ace, & three & two, because I have in order to hold all the same coups as if I had brought forth on the first coup five & ace, & beyond that expectation to bring forth on the second coup five & ace.

(9) One coup which gives me $\frac{8}{36}$, namely carme, because I will have in order to hold at the second coup two & ace, bezet, six & ace, six two & sonnez.

(10) Three coups which give me $\frac{27}{36}$, namely three & ace & double two, in order to hold at the second coup, five & four, five & three, four & three, quine, carme excepted.

(11) Three coups which give to me $\frac{32}{36}$, namely two & ace, because I will have all favorable coups in order to hold, quine, carme, & five & four excepted.

(12) One coup which gives me $\frac{35}{36}$, it is bezet, because there will be at the second coup only quine against me.

The sought lot will be therefore $\frac{565}{1296}$, & the fair division of the wager would be 565 against 731. One would have advantage to wager 3 against 4, & disadvantage to wager 4 against 5.

FOREWARD

It is impossible in the greater part of the situations where two Players are able to be found at Trictrac, to determine what is their lot, & to estimate with precision on what side is the advantage, because beyond the prodigious variety of the different possible dispositions of the thirty dames, the manner often arbitrary by which the Players conduct their game, is that which decides near all the gain of the game. Now all that which depends on the fantasy of the men having no fixed & certain rule, it is clear that one is able to resolve no question on Trictrac, at least when the manner of play is not determined. The only Problem that one is able to resolve in a general manner on the game of Trictrac is the one here: *To find the lot of two Players who are in the jan of return, whatever number dames that they have yet to pass, in some end that they are found placed.* I give here an Example, which will suffice in order to make known in some manner one could find the other more composite cases.

PROBLEM III.
PROPOSITION XXVIII.

Pierre has the three dames A, B, C to raise, & Paul the three dames D, E, F; the one who will have first raised by passing all his dames, will win. One supposes that it is to Pierre to play, one demands what is the advantage of Paul.

When Pierre goes to play, he has twenty-five coups in order to pass the two most remote B & C, eight in order to pass the dames A & C, namely six & ace, five & ace, four & ace, three & ace; two coups in order to pass the dames A & B, & one coup only in order to pass B, namely bezet.

Let S be named the lot of Pierre when he goes to play, x his lot when he brings forth two & ace, & y his lot when he brings forth on the first coup bezet. One will have $S = \frac{33A+2x+y}{35}$. The money of the game is called A.

The concern presently is to determine the unknowns x & y; in order to arrive to the end of it, it is necessary to note that Pierre having no longer to raise but the dame C, is able neither to lose nor to win by the coup that Paul will win; but that his lot will be different according to all the different coups that Paul will bring forth. Because, for example, Paul passing on the first coup the two dames E & F, if Pierre not pass the dame C on his second coup, he will have certainly lost, instead that he would be able yet to win if Paul had passed on his first coup only the dames E & D, or only the dame E.

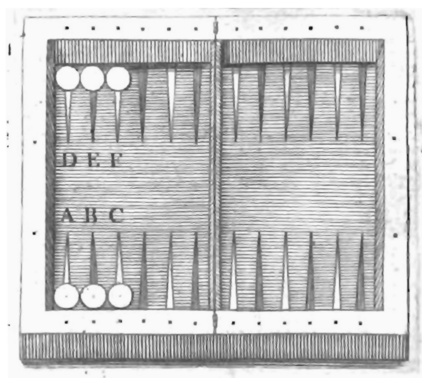
Let therefore u be named the lot of Pierre, when having brought forth on the first coup two & ace, Paul has passed on his coup the dames E & F; h his lot, when Paul has passed the dames D & E; & t his lot, when Paul has passed the dame E. One will have $x = \frac{33u+2h+t}{36}$.

In order to know the value of u, one will note that Pierre having no more than the dame C to pass, he has by playing his second coup, thirty-five coups in order to win.

In order to know the value of h, one will observe that Pierre having more than the dame C to pass, & Paul having no more than the dame F, Pierre has by playing anew, thirty-five coups in order to win, & one coup in order to have $\frac{1}{36}A$: because suppose that Pierre playing for the second time, brings forth bezet which is the only coup which is able to prevent him from winning, Paul has not for that won, he will be able to bring forth also bezet, in which case Pierre would have won.

In order to know the value of t, one will take care that Pierre having more than the dame C, & Paul the two dames D & F to raise, Pierre has by playing for the second time, thirty-five coups in order to win, & one coup in order to have $\frac{4}{36}A$: because Pierre not winning on his second coup, Paul has similarly four coups in order to not raise all his dames, namely bezet, double two, two & ace. One will have therefore $t = \frac{35}{36}A + \frac{4}{36\times36}A$. Having thus determined the unknowns u, h, t, if one substitutes the values found in the equation $x = \frac{33u+2h+t}{36}$, one will have $x = \frac{45366}{46656}A$.

Presently it is necessary to determine the value of y.



Let q be named the lot of Pierre when he goes to play his second coup, & that there remains to him the dames A & C to raise, & to Paul the dame D alone; p his lot when there remains to him to raise the dames A & C, & to Paul the dame F; n his lot when there remains to him to raise the dames A & C, & to Paul the dames D & F. One will have $y = \frac{33q+2p+n}{36}$.

One will find, by some reasoning similar to those that one has made in order to find the value of x, $q = \frac{32}{36}A$, $p = \frac{32}{36}A + \frac{4}{36\times36}A$, $n = \frac{32}{36}A + \frac{4\times4}{36\times36}A$; & consequently $y = \frac{41496}{46656}A$. Having thus determined the values of x & of y, one will find $S = \frac{46641}{46656}A$.

PROBLEM
ON THE GAME OF THE THREE DICE
EXPLICATION OF THE RULES OF THE GAME.

Although this game is ancient & in use in the Académies des Jeux, it is scarcely known but by Players of profession. I believe therefore I must explain with care all the conditions of it.

One will name Pierre the one who holds the die, & Paul will represent the other Players, of whom the number is indefinite, thus as in the Games of *Hazard* & of *Quinquenove*.

Pierre will cast the die until he brings forth either 8, or 9, or 10, or 11, or 12, or 13, that of these chances that Pierre will bring forth will be that which one names the *droit* [right] chance, & will be nearly for him, that which is in the game of the Dupe, for the Player who has the hand, the card that he is given. Next Pierre casts the die: here is in order the principal rules.

(1) The *right* chance being either 9, or 10, or 11, or 12, Pierre will win on the second coup if he brings forth similar chance, that is 9 if the *right* chance is 9, 10 if the *right* chance is 10, &c. He will win also by bringing forth quinze; but he will lose if he brings forth either 3, or 4, or 5, or 6, or 15, or 17, or 18.

(2) If the *right* chance is either 8, or 13, Pierre will win on the second coup by bringing forth it or similar chance, or 16; & he will lose if he brings forth either 3, or 4, or 5, or 6, or 15, or 17, or 18.

(3) In each other case than the preceding two, the number that Pierre will bring forth, after having drawn the *right* chance, will be a chance for the first masse. There is therefore for the masses two chances more than for the *right*, namely 7 & 14.

(4) These two chances being given, Pierre will continue to cast the die, & he will win the first masse, if he brings forth the chance before bringing forth the right; & on the contrary he will lose if he brings forth the right before bringing forth this chance. In the first case the game recommences, & Pierre delivers anew a right & a chance to the first masse, provided nevertheless that he has not *tingued*.

In order to understand that which it is to *tingue*, it is necessary to know that in this game, as in Quinquenove, the Players are able to make the masses, & that Pierre accepts them if he wishes, by saying *Taupe*. But there is here to note, that if Pierre accepting a masse, says *Taupe & tingue*, the first masse goes no further; so that if Pierre brings forth the right after having tingued, he loses all the masses which have been accepted, with the exception of the first which does not go, & he draws that which one just massed. In this case the right & the chance of the first masse subsists; & if Pierre after having tingued brings forth the chance of the first masse, all the masses become null, with the exception of the right & the first masse which subsists.

(5) All the time that Pierre loses the first masse, the one who serves the die to Pierre is able to compel him to hold the paroli; & if that arrives a second time, to hold the seven & the *va*, & next the quinze & the *va*, &c.

(6) When Pierre loses the first masse, one fixes to him either 8, or 9, or 10, or 11, or 12, or 13, but he is obliged to hold only 8 or 13.

In order to make understood perfectly all these rules, I believe that it is proper to apply them to an Example. Let us suppose therefore the right is 13, the first masse 9, the second 10, the third 11, thereupon I make a masse. Pierre says, *Taupe*, & casing the die I bring forth 13. Here is that which will happen; (1) he will win that which just came to be masse; (2) he will lose all the other masses, & I will fix 13 to him by making, if I wish, the paroli of this masse, next he will be given one chance.

If Pierre instead of saying simply *Taupe*, had said: *Taupe & tingue*, all would have been as above, with this sole difference that he had not lost the first masse at all, & that it would be remained as well as the right.

Let us suppose now that Pierre brings forth 9 after having said *Taupe*, he will draw the first masse which is 9, all the other masses will be had, & Pierre will recommence the game by drawing a right at random. If he had said, *Taupe & tinque*, Pierre would have neither lost, nor won, & all the chances had been null, with the exception of the first which will substitute with the right. Finally when Pierre will bring forth either 10, or 11 before bringing forth 13, he will win that of these masses which he will bring forth before the right.

PROBLEM
PROPOSITION XXIX.

One demands what is in this Game the advantage or the disadvantage of the one who holds the die.

Let x be the lot of Pierre when he brings forth for right chance 8 or 13, y his lot when he brings forth either 9 or 12, & z when he brings forth 10 or 11.

S will express the lot of Pierre, & A the stake of Paul. One will have $S = \frac{21x+25y+27z}{73}$. One will find also

$$x = \frac{100A + 50 \times \frac{25}{23}A + 27 \times \frac{9}{4}A}{216} = \frac{19789}{19872}A.$$
$$y = \frac{95A + 42 \times \frac{21}{23}A + 27 \times \frac{17}{13}A + 15 \times \frac{3}{2}A}{216} = \frac{126731}{129168}A.$$
$$z = \frac{101A + 21 \times \frac{7}{4}A + \frac{25\times25}{13}A + 30 \times \frac{5}{7}A}{216} = \frac{15147}{26208}A.$$

Consequently the disadvantage of Pierre will be expressed by this quantity,

$$\frac{21 \times \frac{83}{19872}A + 25 \times \frac{2437}{129168}A + 27 \times \frac{3183}{78624}A}{73},$$

which is reduced to this fraction $\frac{1494103}{66004848}A$ which is greater than $\frac{1}{45}$, & smaller than $\frac{1}{44}$, & this would be thence the sought disadvantage, if one supposed that Pierre must quit the die & finish the game as soon as he would have either won or lost. Thus the first masse being a pistole, there is out of this sum 4 sols 7 den. of loss for Pierre when he must draw his right at random. But when one has fixed 8 or 13 to him, his disadvantage with respect to the first masse, is only 10 d. $\frac{5}{207}$. One will see in the Remarks which follow what is his disadvantage in accepting the masses.

REMARK I.

It is less disadvantageous to Pierre to have 8 or 13 for right chance, than to have 9 or 12; & it is less disadvantageous to him to have 9 or 12, than to have 10 or 11: because I find that the right chance being 8 or 13, the disadvantage of Pierre with respect to the stake of Paul, is greater than $\frac{1}{240}A$, & smaller than $\frac{1}{239}A$: That the right chance being either 9 or 12, the disadvantage of Pierre is greater than $\frac{1}{54}A$, & less than $\frac{1}{53}A$; & finally that the right chance being 11 or 10, his disadvantage is greater than $\frac{1}{25}$, & smaller than $\frac{1}{24}$.

REMARK II.

To be able to tinguer is a privilege that this Game accords to the one who holds the dice box, by which he is master to make endure a long time the right & the first masse. It is easy to perceive that this advantage is not very considerable, & holds only when the chance of the right must happen more often than the chance of the first masse; for example when the right being 10 or 11, the first masse is 8 or 13. In this case it is worth more to tingue than to taupe simply; but it would be yet more proper to not accept masse.

REMARK III.

There is in this game no circumstance where the one who holds the dice box has the advantage over the Players. Here is the rule that he must follow in order that his disadvantage be the least that it will be possible. He will not accept at all masses when the right will be 9 or 12, & again less when it will be 10 or 11: because in the first case he has on one masse of one pistole three sols nine deniers of loss, & in the second eight sols two deniers.

REMARK IV.

One sees by the preceding observations, that there is lacking much in it that this Game is neither as equal, nor as well invented as well as the Players imagine it. In order to reform it it would be proper to rule that 17 was as well as 15 a chance favorable to the one who holds the dice box, whether the right is either 9, or 10, or 11, or 12. By this reform the disadvantage of Pierre that one has found $= \frac{1494103}{66004848}A$, will be expressed by this fraction $\frac{188071}{66004848}A$, that which is worth a little less than seven deniers, A designating a pistole.

DEFINITION

I will call simple dice the dice of different kind, or which mark different points; double die, two dice of same kind, or which the same points mark, for example, double two, ternes, &c. triple die, three dice of same kind, for example, three aces or three twos, &c. & thus of quadruple, quintuple, sextuple, &c. for or five or six dice of same kind.

PROBLEM
PROPOSITION XXX.

Let there be any number of dice, Pierre wagers that casting them at random, he will bring forth as many of them of one kind, as many of another; for example, so many singles, so many doubles, so many triples, or so many doubles, so many quadruples, &c. One demands what will be his lot, & how many different ways he will have to bring forth the dice in a manner that he will have proposed it.

Let p be the number of dice, q the number of all the diverse possible arrangements of these dice; b the exponent of the die which has the highest dimension among all those that Pierre is proposed to bring forth; c, d, e, f, &c. the exponent of the other dice that he must bring forth, of which the exponent must be less, so that c expresses a number smaller than b, & d a number smaller than c, & e a number smaller than d, & f a number smaller than e, &c.

I will name also B the number of these dice that one demands of the dimension expressed by b, C the number of the dice that one demands of the dimension expressed by c, D the number of the dice that one demands of the dimension expressed by d, E the number of the dice that one demands of the dimension expressed by e, F the number of the dice that one demands of the dimension expressed by f, &c.

I will call further k, l, m, n, r, &c. the numbers which express all the diverse possible arrangements of the numbers designated by the letters b, c, d, e, f, &c. I will express also by this mark $\underset{n}{\overset{6}{\square}}$ the number which expresses in how many ways B is able to be taken in 6, setting the smallest number below, & the greatest above, & between two this arbitrary mark $\square$, thus as in proposition XIV.

All this supposed, the lot of Pierre will be expressed by a fraction, of which the numerator will be q multiplied by this series of products,

$$\underset{B}{\overset{6}{\square}} \times \underset{C}{\overset{6-B}{\square}} \times \underset{D}{\overset{6-B-C}{\square}} \times \underset{E}{\overset{6-B-C-D}{\square}} \times \underset{F}{\overset{6-B-C-D-E}{\square}} \times \&c.$$

& the denominator will be 6^p multiplied by this series of products,

$$B \times k \times C \times l \times D \times m \times E \times n \times F \times r \times \&c.$$

It is necessary to remark, (1) that this formula being applied to a particular case, must be composed, either in the numerator, or in the denominator, only by having products which are marked by $\times$, as there are the different dimensions among the dice that one wishes to bring forth.

(2) That if the dice, instead of having six faces had any number of them expressed by R, there would be only to substitute R instead of 6 in this formula. But it is necessary to observe that then the formula would become an infinite series, R being indeterminate.

Finally in order to have the number of coups favorable to Pierre, there is only to multiply this formula by 6 raised to the exponent p. I will not give at all the demonstration of this formula, because it would be only extremely long & abstract, & would be understood only by those who will be themselves capable of finding it.

I will be content to give here a quite extended Table, which will facilitate the comprehension, & which will uncover in part the usage of this Problem. This Table gives all the different cases of the Problem proposed from two dice to nine inclusively. The first column will give all the determinate cases, that which makes a particular kind of the general Problem: the second gives them indeterminate conformable to the enunciation of the Problem. Thus, for example, one will find that the number 3 expresses how many different ways there are to bring forth bezet & a two with three dice, & the number 90 which is opposite in the second column, how many different ways there are to bring forth any double with any single also; & likewise the number 12 will express how many different ways there are to bring forth bezet, a two & a three with four dice, & the number 720, how many there are to bring forth a double & two singles.

TABLE

For two dice.

		Determinate		Indeterminate	
(1)	in order to have two singles,	2	there are	30	coups.
(2)	a doublet,	1		6	

For three dice.

		Determinate		Indeterminate	
(1)	in order to have three singles,	6		110	
(2)	a double & a single,	3	there are	90	coups.
(3)	a triple,	1		6	

For four dice.

		Determinate		Indeterminate	
(1)	in order to have four singles,	24		360	
(2)	a double & two singles,	12		720	
(3)	two doubles,	6	there are	90	coups.
(4)	a triple & a single,	4		120	
(5)	a quadruple,	1		6	

For five dice.

		Determinate		Indeterminate	
(1)	in order to have five single dice,	120		720	
(2)	a double & three singles,	60		3600	
(3)	two doubles & one single,	30		1800	
(4)	a triple & two singles,	20	there are	1200	coups.
(5)	a triple & a double,	10		300	
(6)	a quadruple & a single,	5		150	
(7)	a quintuple,	1		6	

For six dice.

		Determinate		Indeterminate	
(1)	in order to have six singles,	720	there are	720	coups.
(2)	a double & four singles,	360		10800	
(3)	two doubles & two singles,	180		16200	
(4)	three doubles,	90		1800	
(5)	a triple & three singles,	120		7200	
(6)	a triple, a double & a single,	60		7200	
(7)	two triples,	20		300	
(8)	a quadruple & two singles,	30		1800	
(9)	a quadruple & a double,	15		450	
(10)	a quintuple & a single,	10		180	
(11)	a sextuple,	1		6	

For seven dice.

(1)	in order to have a double & five singles,	2520	there are	15120	coups.
(2)	two doubles & three singles,	1260		75600	
(3)	three doubles & a single,	630		37800	
(4)	a triple & four singles,	840		25200	
(5)	a triple, a double & two singles,	420		75600	
(6)	a triple & two doubles,	210		12600	
(7)	two triples & a single,	140		8400	
(8)	a quadruple & three singles,	210		12600	
(9)	a quadruple, a double & a single,	105		12600	
(10)	a quadruple & a triple,	35		1050	
(11)	a quintuple & two singles,	42		2520	
(12)	a quintuple & a double,	21		630	
(13)	a sextuple & a single,	7		210	
(14)	a sextuple,	1		6	

For eight dice.

		Determinate		Indeterminate	
(1)	in order to have two doubles & four singles,	10080	there are	151200	coups.
(2)	three doubles & two singles,	5040		302400	
(3)	four doubles,	2520		37800	
(4)	a triple & five singles,	6720		40320	
(5)	a triple, a double & three singles,	3360		403200	
(6)	a triple, two doubles, & a single,	1680		302400	
(7)	two triples & two singles,	1120		100800	
(8)	two triples & a double,	560		33600	
(9)	a quadruple & four singles,	1680		50400	
(10)	a quadruple, a double & two singles,	840		151200	
(11)	a quadruple & two doubles,	420		25200	
(12)	a quadruple, a triple & a single,	280		33600	
(13)	two quadruples,	70		1050	
(14)	a quintuple & three singles,	336		20160	
(15)	a quintuple, a double & a single,	168		20160	
(16)	a quintuple & a triple,	56		1680	
(17)	a sextuple & two singles,	56		3360	
(18)	a sextuple & a double,	28		840	
(19)	a sextuple & a single,	8		240	
(20)	an octuple,	1		6	

For nine dice.

		Determinate		Indeterminate	
(1)	in order to have three doubles & three singles,	45360	there are	907200	coups.
(2)	four doubles & a single,	22680		680400	
(3)	a triple, a double & four singles,	30240		907200	
(4)	a triple, two doubles & two singles,	15120		2721600	
(5)	a triple & three doubles,	7560		453600	
(6)	two triples & three singles,	10080		604800	
(7)	two triples, a double & a single,	5040		907200	
(8)	three triples,	1680		33600	
(9)	a quadruple & five singles,	15120		90720	
(10)	a quadruple, a double & three singles,	7560		907200	
(11)	a quadruple, two doubles & a single,	3780		680400	
(12)	a quadruple, a triple & two singles,	2520		453600	
(13)	a quadruple, a triple & a double,	1260		151200	
(14)	two quadruples & a single,	630		37800	
(15)	a quintuple & four singles,	3024		90720	
(16)	a quintuple, a double & two singles,	1512		272160	
(17)	a quintuple & two doubles,	756		45360	
(18)	a quintuple, a triple & a single,	504		60480	
(19)	a quintuple & a quadruple,	126		3780	
(20)	a sextuple & three singles,	504		30240	
(21)	a sextuple, a double & a single,	252		30240	
(22)	a sextuple & a triple,	84		2520	
(23)	a sextuple & a two singles,	72		4320	
(24)	a sextuple & a double,	36		1080	
(25)	an octuple & a single,	9		270	
(26)	a nontuple,	1		6	

REMARK.

If the games of dice are in such small number, & are played only with two dice, or at most with three, with the difference from the games of cards which are played with a quite great number of cards, there is also the appearance that comes from this that one has not been able to calculate the chances which are found among many dice. In fact this was quite difficult. The preceding Table, & those that one will find in the propositions which follow to the end of this second Part, will give thence above all the light that one will be able to wish, & will serve to those who would wish to invent some more varied games of dice, & consequently more agreeable than all those that one has known to the present.

PROBLEM
PROPOSITION XXXI.

On demands in how many ways one is able to bring forth a certain number of determined points, with a certain number of dice.

All the Players of Trictrac know in how many ways each point from two to twelve, are able to be brought forth. Mr. Huygens has given a Table of it for two & for three dice: but one is not able to go further without method, because that becomes immediately extremely composed. One will find by examining the solution of the preceding Problem, that the one here is contained there. Here is a Table which determines all these chances from two to nine inclusively.

TABLE

With two dice.

There is	coups that give	
1		2 or 12
2		3 or 11
3		4 or 10
4		5 or 9
5		6 or 8
6		7

With three dice.

There is	coups that give	
1		3 or 18
3		4 or 17
6		5 or 16
10		6 or 15
15		7 or 14
21		8 or 13
25		9 or 12
27		10 or 11

With four dice.

There is	coups that give	
1		4 or 24
4		5 or 23
10		6 or 22
35		7 or 21
56		8 or 20
80		9 or 19
104		10 or 18
125		11 or 17
140		12 or 16
146		13 or 15
		14

With five dice.

There is	coups that give	
1		5 or 30
5		6 or 29
15		7 or 28
35		8 or 27
70		9 or 26
126		10 or 25
205		11 or 24
305		12 or 23
360		13 or 22
480		14 or 21
561		15 or 20
795		16 or 19
930		17 or 18

With six dice.

There is	1	coups that give	6 or 36
	6		7 or 35
	21		8 or 34
	56		9 or 33
	126		10 or 32
	252		11 or 31
	456		12 or 30
	756		13 or 29
	1161		14 or 28
	1666		15 or 27
	2247		16 or 26
	2856		17 or 25
	3431		18 or 24
	3906		19 or 23
	4221		20 or 22
	4332		21

With seven dice.

There is	1	coups that give	7 or 42
	7		8 or 41
	28		9 or 40
	84		10 or 39
	210		11 or 38
	462		12 or 37
	917		13 or 36
	1667		14 or 35
	2807		15 or 34
	4417		16 or 33
	6538		17 or 32
	9142		18 or 31
	12117		19 or 30
	15267		20 or 29
	18327		21 or 28
	20993		22 or 27
	22967		23 or 26
	24017		24 or 25

With eight dice.

There is	coups that give
1	8 or 48
8	9 or 47
36	10 or 46
120	11 or 45
330	12 or 44
792	13 or 43
1708	14 or 42
3368	15 or 41
6147	16 or 40
10480	17 or 39
16808	18 or 38
25488	19 or 37
36688	20 or 36
50288	21 or 35
65808	22 or 34
82384	23 or 33
98813	24 or 32
113688	25 or 31
125588	26 or 30
133288	27 or 29
135954	21

With nine dice.

There is	coups that give
1	9 or 54
9	10 or 53
45	11 or 52
165	12 or 51
495	13 or 50
1287	14 or 49
2994	15 or 48
6354	16 or 47
12465	17 or 46
22825	18 or 45
39303	19 or 44
63999	20 or 43
98979	21 or 42
145899	22 or 41
205560	23 or 40
277464	24 or 39
359469	25 or 38
447669	26 or 37
536569	27 or 36
619569	28 or 35
689715	29 or 34
740619	30 or 33
767394	31 or 32

One will find by this Table that eleven, for example, is brought forth in two ways with two dice, in 27 ways with three dice, in 104 ways with four dice, in 205 ways with five dice, &c.

COROLLARY

One is able with neither advantage nor disadvantage to play with three dice at pass-ten, & with five dice at pass-seventeen, & with seven dice at pass-twenty-four, & thus consecutively, by adding always 7. But it is necessary to remark that the number of dice being even, one is not able to make equal divisions, since there is always a certain point that one is able to bring forth rather than each other: with two dice, it is 7; with four dice, it is 14; with six dice, it is 21, &c. by adding always 7.

REMARK.

It is necessary to observe that the Players have established so much for the game of Rafle as for Pass-ten, that there would be good coups only those where there would be at least two similar dice. I have not been able to guess that which has occasioned this rule which serves to amuse the Players, since there is at each coup odds of five against four that the coup that one is going to play will not be good; & I would believe that one would not do ill to abolish it, by establishing that all the coups make good, or if one wishes (by reversing the ordinary rule) that those there alone are reputed for good where all the dice would mark different points; thus one would have fewer of these useless coups which annoy near always both the Players & the Spectators. For the rest with either of these changes Pass-ten would be always an equal game. The preceding Table proves it under the supposition that each coup is reputed good. I will show in the following that this game would be yet equal, by supposing that it had good coups only those where all the dice would be different, or else, according to the ordinary rule of this game, that it has good only those where there are found at least two similar dice.

PROBLEM
PROPOSITION XXXII.

One demands how many different coups one is able to bring forth with a certain number of dice given at will.

It is necessary to remark, (1) that each die having six faces, two dice produce necessarily thirty-six coups, & three dice two hundred sixteen coups, that which is the cube of six, & four dice twelve hundred ninety-six coups, that which is the fourth power of six. (2) That in the thirty-six coups that two dice give, there are six which are able to arrive only in one way, namely the six doublets, & that there are fifteen, namely 6 & ace, 6 & 2, 6 & 3, 6 & 4, 6 & 5; 5 & ace, 5 & 2, 5 & 3, 5 & 4; 4 & ace, 4 & 2, 4 & 3; 3 & ace, 3 & 2; 2 & ace, which each are able to arrive in two ways, because the one of the two dice which has brought forth an ace, the other die being a six, is able to be a six, the other being an ace, & thus of the others. It is therefore certain that there are only twenty-one different coups in two dice, although really there are thirty-six coups in two dice.

One is able to note the same thing for three dice, for example, ace, ace & 2 is able to arrive in three ways, because each of the three dice will be able to be a two, the other two being aces; & likewise ace, 2, 3 is able to arrive in six ways, because one of the three dice marking an ace, each of two others is able to be either a two or a three; & one of the three dice being a two, each of the two others is able to be either an ace or a three; & finally one of the three dice being a three, each of the two others is able to be either an ace or a two. One sees therefore that if in the two hundred sixteen possible coups of three dice one wishes to count ace, 2, 3, & ace, ace, 2 & each of the others of this kind, only for one coup, that is, to count only once all those which arrive either in three or in six ways; this number of two hundred sixteen reduces to the sole coups which are different from one another, will be much less: the concern is to find a method to determine this number of coups different from one another for such number of dice as there be. Here is one very general & very brief which depends on proposition 30.

Let $p = 6$: one will have the sought number of coups for one die $= p$, for two dice $= p \times \frac{p+1}{2}$, for three dice $= p \times \frac{p+1}{2} \times \frac{p+2}{3}$, for four dice $= p \times \frac{p+1}{2} \times \frac{p+2}{3} \times \frac{p+3}{4}$, for five dice $= p \times \frac{p+1}{2} \times \frac{p+2}{3} \times \frac{p+3}{4} \times \frac{p+4}{5} \times$ &c., so that with two dice one will have twenty-one different coups, fifty-six with three dice, one hundred twenty-six with four dice, two hundred fifty-two with five dice, four hundred sixty-two with six dice, seven hundred ninety-two with seven dice, twelve hundred eighty-seven with eight dice, &c.

One will be able to note that these numbers 6, 21, 56, 126, 252, 462, 792, 1287, & the other following compose the sixth transversal band of the Table, page 54, continued at discretion.

PROBLEM
PROPOSITION XXXIII.

Pierre plays against Paul at Pass-ten & holds the die, Paul proposes to him to give to him a point on condition that that ace which he gives will serve to render the good coups when Pierre will bring forth an ace from one of his three dice. One demands if Pierre must accept this game.

The reason for doubt is that if this fourth die which bears an ace, gives to Pierre his favorable coups, which without that had been contrary or indifferent, there are many also among them which would be indifferent, or even favorable to Pierre, which become contrary to him.

In order to resolve this question, it is necessary to consult the Table of Proposition 30 for three dice. One will note, (1) that there are forty-eight coups which make Pierre win independent of this fourth die; (2) that there are twenty-four of them which had made him recommence, & which by means of this new ace make him win. These twenty-four coups are, 6, 4, 1; 6, 5, 1; 6, 3, 1; 5, 4, 1; (3) that there are nine which make him win, & which without this fourth die had made him lose. These nine coups are 4, 4, 2; 4, 3, 3; 6, 2, 2; (4) that there are thirty-nine which make him lose independently of the fourth die, & thirty-six which make him lose because of this new ace, & which without that were indifferent. These thirty-six coups are 1, 4, 2; 1, 2, 3; 1, 2, 5; 1, 2, 6; 1, 3, 4; 1, 3, 5; so that there remains sixty indifferent coups, namely 4, 3, 2; 5, 3, 2; 6, 3, 2; 5, 4, 2; 6, 4, 2; 5, 4, 3; 6, 4, 3; 6, 5, 2. There would be therefore in this division the advantage for Pierre, but it would be only the fifty-second part of the money set into the game.

One is able to observe that if one counted a point to the profit of Pierre only when the ace, represented by the fourth die, serves to render good a coup which had been null, the game would be disadvantageous to Pierre, & his disadvantage would be precisely the quadruple of that which is his advantage under the preceding supposition.

This Problem is, as one sees, quite easy, & I have set it here only because it has been proposed to me by one of my friends, who has said to me to have often seen a playing end-to-end following the conditions that one has explicated under the enunciation of the Problem.

PROBLEM I.
ON THE GAME OF RAFLE.
PROPOSITION XXXIV.

Pierre plays at the first Rafle with a certain number of Players at will. One demands what will be his advantage when he will have any points from eleven to eighteen.

There are two types of games of Rafle, namely the first Rafle, & the three counted Rafles. I am going to give here that which regards the first Rafle, the following Problem will be on the three Rafles counted. Here are some rules common to these two games. (1) One plays with three dice. (2) All the coups where there is not found at least two similar dice are reputed nulls, & one recommences them. (3) In these games there is no primacy at all, & when two or many Players are found to have the same point, they recommence among them in order to see who will win. Here are some other rules which are particular to the game of the first Rafle. (1) A Player says that he has rafle when the three dice that he has cast bear all the same point. (2) Rafle carries it away on those who have only some points, so that, for example, only the one who will have rafle will win by prejudice from the one who will have 17; beyond this case the one who has the highest point wins. (3) A higher rafle carries it away over a lower base, for example rafle of 4 over rafle of 3, & rafle of three over rafle of 2, &c.

The solution of this Problem will be understood easily by an Example.

I suppose therefore that there are three Players, Pierre, Paul & Jacques: Pierre has already played & has brought forth eleven. One demands if there is advantage, & what is this advantage.

It is necessary first to consult proposition 30 in order to find how many good coups there are in three dice, that is coups where there is found at least two similar dice, & how many of these coups there are in order to bring forth each of the different points in particular: next it is necessary to employ the analytic method, to examine by order that which is able to arrive in the coups of Paul & of Jacques, & that which the different hazards of these two coups give to Pierre in expectation, either of gain, or of loss.

I find that there are three coups in order to bring forth 17 or 4, six coups in order to bring forth 16 or 5, four coups in order to bring forth 15 or 6, nine coups in order to bring forth 14 or 7, 13 or 8, 10 or 11, & finally seven coups in order to bring forth 12 or 9. This supposed, here is how I reason.

When Paul will play his coup, Pierre will lose if Paul brings forth either 18 or 17, either 16 or 15, either 14 or 13, either 12 or rafle of aces, of two & of three, that which makes forty-two coups in order to lose. There are nine coups in order that Pierre is end-to-end with Paul in the awaiting the coup of Jacques, & forty-five coups in order that Paul bringing forth any point below eleven, Pierre has no more to fear than the coup of Jacques.

When Paul has brought forth any point below eleven, the lot of Pierre is to have forty-five coups in order to win all that which is in the game, & nine coups in order to divide with Jacques, namely when Jacques brings forth eleven.

If Paul has brought forth eleven, the lot of Pierre is to have forty-five in order to divide equally with Paul the right out of all that which is in the game; nine coups in order to have his third out of the money which is in the game; & finally to have forty-two coups to lose.

If one reduces this reasoning according to the rules of algebra, one will find that the sought lot of Pierre is $\frac{819}{1024}A$, by supposing that A expresses the stake of each Player, that which shows that Pierre has disadvantage when playing with two Players he has eleven. This disadvantage is such that he would be able to have with neither loss nor profit to give forty sols & a fraction of deniers to a Player who would wish to take his place, supposed that A which designates the stake of each Player expresses a pistole.

One will be able to find in this manner the advantage or the disadvantage of Pierre, whatever be his point, & whatever number of Players that there be. Here is a Table of it which gives the advantage of Pierre, by supposing that he has any point from eleven to eighteen, other than by a rafle. One supposes, as above, that the game is to the pistoles.

One sees by this Table that between two Players there is advantage to have eleven, & that with three Players there is disadvantage.

When there are four Players, one has advantage only when one has at least thirteen. I find that with twelve points there is out of one pistole one livre twelve sols of loss or disadvantage, that which appears first rather strange.

I have found some persons of mind who believe to see evidently that since there is advantage between two Players to have eleven points, one must conclude that there would be also an advantage such number of Players as there were. Here is how they reason. It is true that Pierre playing third, & having eleven points, has half less expectation to win, than when having eleven points, there is concern only to one Player; but in recompense he has the double to win. Now the product of $2 \times \frac{1}{2}$ being $= 1$, it follows that Pierre having eleven points, must have advantage, if the game is among three Players, or if it is only between two Players. They employed the same reasoning in order to prove that the advantage of the one who has eleven points is the same, if there be only two Players, or if there be four, or any other number.

TABLE

	For two Players.			*For three Players.*			*For four Players.*		
	advantage			advantage			advantage		
points	liv.	sols,	den.	liv.	sols,	den.	liv.	sols,	den.
18	9	17	11	19	13	$9\frac{25}{96}$	29	8	$4\frac{585}{82944}$
17	8	8	9	15	9	$10\frac{71}{128}$	21	6	$11\frac{1119}{3072}$
16	7	10	0	12	19	$3\frac{21}{64}$	14	7	$1\frac{251}{256}$
15	6	11	3	10	11	$6\frac{9}{32}$	12	14	$5\frac{3009}{3072}$
14	5	6	3	7	12	$1\frac{25}{32}$	8	0	$4\frac{599}{1024}$
13	3	8	9	3	11	$3\frac{15}{32}$	2	3	$9\frac{825}{1024}$
12	1	17	6	0	11	$8\frac{5}{8}$			
11		6	3						

This reasoning is specious, but there is lacking in this that one supposes that the expectation that Pierre has to win is half less when two Players have to play after him, than when there is only one of them: that which is not true at all, although quite possible. One is not able to seek too much evidence in this matter, where one will find more than each other, that the appearances lead to error.

PROBLEM II.
ON THE GAME OF THREE RAFLES COUNTED.
PROPOSITION XXXV.

Pierre plays against Paul to whom will make the most points in three rafles counted, that is, in three coups such that there is found at least one doublet in the three dice. He has brought forth thirty-two. One demands if there is advantage, & what is this advantage.

One would be able to resolve this Problem by analysis, by examining in order all the different points that one is able to bring forth with one, two, three, four, &c. to nine dice, & by rejecting all those where in each three dice there would be found three dice different from one another, & by expressing all these different chances by some unknowns that one would determine according to the ordinary rules: but this way would be excessively long, & would demand a calculation of many months. The Problem of Proposition 30 furnishes one very abridged. Here is a Table which contains all the different chances which are able to arrive, & to express the advantage of Pierre for all the different points that there will be from 32 to 54.

This Table is, as one sees, arranged on four columns. The first designates all the different points that Pierre is able to have from nine to fifty-four. the second expresses the number of different coups that are able to give the points which correspond to it in the first column. The third column gives the advantage of Pierre for all the different points that he is able to have from thirty-two to fifty-four, by giving to each of these terms the quantity 1769472 for the denominator. The fourth column gives this advantage in livres & in sols, by supposing that the game is in pistoles, that is, that Pierre has set one pistole into the game. One has neglected the deniers.

TABLE.

points	diverse ways to bring them	advantage	livres	sols.
54 or 9	1	884735	9	19
53 or 10	9	884725	9	19
52 or 11	45	884671	9	19
51 or 12	147	884479	9	19
50 or 13	369	883963	9	19
49 or 14	765	882829	9	19
48 or 15	1446	880618	9	19
47 or 16	2484	876688	9	18
46 or 17	3969	870235	9	16
45 or 18	5869	860397	9	14
44 or 19	8433	846095	9	11
43 or 20	11493	826169	9	6

points	diverse ways to bring them	advantage	livres	sols.
42 or 21	15027	799649	9	0
41 or 22	19287	765335	8	13
40 or 23	23886	722162	8	3
39 or 24	28668	669608	7	11
38 or 25	38867	607073	6	17
37 or 26	38871	534335	6	0
36 or 27	43171	452293	5	2
35 or 28	47457	361665	4	1
34 or 29	50607	363601	2	19
33 or 30	52551	160443	1	16
32 or 31	53946	53946	0	12

One sees by this Table that the advantage to have some of the different points from fifty-four to forty-two, goes only to twenty sols difference; & that the one to have some of the numbers from fifty-four to forty-eight, goes only to some deniers.

One sees on the contrary that this difference changes quite considerably in the numbers which approach thirty-two. One is able to uncover by the reasoning that it must be nearly so.

It appears by the comparison of this Table with that of Proposition 31 for nine dice, that one would have more advantage if playing with three dice three coups in sequence, all the coups were good indifferently: because in this case the advantage of a Player which would have for point twenty-two would be $\frac{767394}{10077696}$, that which would be twenty-one sols & some deniers advantage or profit, the game being in pistoles.

PROBLEM
ON THE GAME OF THE SAVAGES,
CALLED
GAME OF THE NUTS.

The Baron of Hontan makes mention of this Game in the second Volume of his Voyages of Canada, *p. 113*. Here is how it is explained:

One plays with eight nuts black on one side & white on the other: one casts the nuts into the air: then if the blacks are found odd, the one who has cast the nuts wins that which the other Player has set into the game: If they are found either all blacks or all whites, he wins the double of them; & beyond these two cases he loses his stake.

PROPOSITION XXXVI.

One demands which of the two Players has the advantage, by supposing that they set equally into the game.

The solution of this Problem depends on Proposition 30, of which it is only a particular case: That which one will recognize if one imagines eight dice which having each two faces, on one of which is marked an ace, & on the other a two. Thus supposed, it will be clear that the Problem of the Nuts is reduced to the one here. To determine how much there is to wager that casting these eight dice at random, one will bring forth either one ace & seven twos, or three aces & five twos, or five aces & three twos, or seven aces & one two, or two aces & six twos, or four aces & four twos, or six aces & double two: that which is found contained in the very general formula of Proposition 30.

One will find that there are, (1) eight coups out of 256 in order to bring forth one black & seven whites; (2) 56 coups in order to have three blacks & five whites; (3) 28 coups in order to have two blacks & six whites; (4) 70 coups in order to have four blacks & four whites. It is evident that one is able to bring them forth either all black or all white only in one way. It follows from that that if the money of the game is called A, the lot of the one who casts the nuts will be $\frac{128\times A+2\times\overline{A+\frac{1}{2}A}}{256}$, & the lot of the other Player will be $\frac{126A+2\times\overline{0-\frac{1}{2}A}}{256}$. Thus the advantage of the one who casts the nuts is $\frac{3}{256}$; & in order that the game may be equal, it would be necessary that the one who casts the nuts set into the game 22 against the other 21.

One is able to observe that the inequality of this game carries no prejudice to those Players of the other world, who playing among them only some things of which the property is common to them, must be rather indifferent for the gain & for the loss. The contempt that these People have for that which we regard more, is a kind of paradox that one must not advance at all without proof in a Book such as the one here. Here is it drawn from Baron de la Hontan: *Moreover*, says this agreeable Voyager, *these games are only for some feasts, & for some other trifles: because it is necessary to note that as they hate money, they never set it on their games. Also one is able to assure that interest has never caused division among them.*

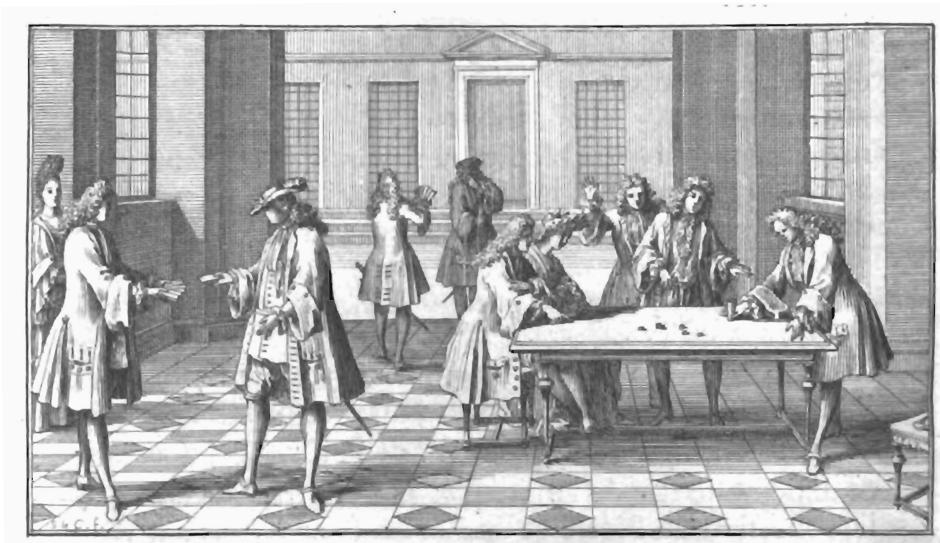
I believe I must add that this Problem has been proposed to me by a Lady, who has given it to me nearly immediately a quite correct solution, by serving herself with the Table page 54: but this Table serves here only by chance, because if the nuts instead of having two faces, had further, for example, four of them, this Table would be not at all sufficient, & the Problem would be much less easy than the preceding, thus as one will be able to note it in the following Problem.

One supposes that the eight nuts have each four faces, namely a white, a black, a green & a red. Pierre will be the one who casts the nuts, Paul will be the other player.

If the nuts having been cast at random, there are found of the four colors, Paul will give B to Pierre. If there are only three colors, Paul will give to him $3B$; & if there is only one color alone, that is, if the eight nuts are either all whites or all blacks, or all greens or all reds, Paul will give to him $4B$; finally if there are only two colors, Pierre will give to Paul $2A$.

This supposed, *One demands on what side is the advantage, & what is this advantage, by supposing that A has to B any ratio.*

One will find by Proposition 30 that if $B = A$, Paul will have the advantage in the game, but it would be only by this fraction $\frac{233}{16384}$, that which is very nearly the sixtieth & tenth part of unity; & consequently so that the condition of Pierre & of Paul make equals, it would be necessary that B was $= \frac{11592}{11359}A$, that is that Pierre must set into the game eleven thousand five hundred fifty-two against Paul eleven thousand three hundred fifty-nine.



THIRD PART
Where one gives the solution of five Problems proposed by Mr. Huygens.

PROBLEM I.
PROPOSITION XXXVII.

Pierre & Paul play together with two dice: Here are the conditions of the game. Pierre will win by bringing forth six, & Paul bringing forth seven. Each of the two will play two coups in sequence when he will have the dice: however Pierre who will commence will play only one for the first time. The concern is to determine the lot of each of these two Players, or the expectation that each will have to win the game.

SOLUTION.

Since each face of one of the dice is able to be found successively with all the faces of the other, it is clear that the two dice are able to give thirty-six coups, & that of these thirty-six coups there are five which give the number of six, namely ace & five, five & ace, two & four, four & two & terne; & six which give the number of seven, namely ace & six, six & ace, two & five, five & two, three & four, four & three.

Presently let A be named the money of the game, x the lot of Pierre when he is going to play his coup, y his lot when Paul is going to play his first coup, z his lot when Paul is going to play his second coup, & finally u his lot when the turn of Pierre returns he is going to play the first of his two coups.

One will have these four equalities, $S = \frac{5}{36}A + \frac{11}{36}y$, $y = \frac{30}{36}z$, $z = \frac{30}{36}u$, $u = \frac{5}{36}A + \frac{31}{36}x$; that which gives $S = \frac{5}{36}A + \frac{31}{36} \times \frac{25}{36} \times \overline{\frac{5}{36}A + \frac{31}{36}x}$; whence one draws by reduction & transposition $S = \frac{10355}{22631}A$, that which expresses the lot of Pierre; & $A - S = \frac{12276}{22631}A$ which expresses the one of Paul.

REMARK.

If one supposed that Pierre played first one coup, & Paul two coups, next Pierre two coups & Paul three coups, next Pierre three coups & Paul four coups, & thus consecutively, Paul playing always one more coup than Pierre, the lot of Pierre would be expressed by an infinite series of which it would be quite difficult to have the sum, this series would be $= \frac{b}{f}A + \frac{d\times c^2\times b}{f^4}A + \frac{d^2\times c^2\times b}{f^5}A + \frac{d^3\times c^5\times b}{f^9}A + \frac{d^4\times c^5\times b}{f^{10}}A + \frac{d^5\times c^5\times b}{f^{11}}A + \frac{d^6\times c^9\times b}{f^{16}}A + \frac{d^7\times c^9\times b}{f^{17}}A + \frac{d^8\times c^9\times b}{f^{18}}A + \frac{d^9\times c^9\times b}{f^{19}}A + \frac{d^{10}\times c^{14}\times b}{f^{25}}A +$ &c. by supposing $b = 5$, $c = 30$, $d = 31$, $f = 36$.

It is easy to note the order of the series, & to continue it to infinity, the expression of the lot of Paul would be the quantity which is lacking to the series which expresses the lot of Pierre in order to be worth A.

If one supposed that Pierre & Paul play with a die, according to the order that one just noted, to which the first would bring forth a six, one would have for expression of the lot of Pierre a simpler series, namely $1 - p - p^3 - p^5 + p^8 - p^{12} + p^{15} - p^{19} + p^{14} - p^{19} +$ &c. by supposing that p is $= \frac{5}{6}$.

PROBLEM II.
PROPOSITION XXXVIII.

Three Players, Pierre, Paul & Jacques play together, & agree that drawing one after the other a token at random among twelve, of which eight will be blacks & four will be whites, the one who first will have drawn a white token will win. Here is the order according to which they play: Pierre draws first, Paul draws second, & Jacques third; next Pierre recommences, & the others follow him according to their rank, until one of the Players has won. The concern is to find that which each Player must set into the game, so that the game is equal; or else, that which reverts to the same, the concern is to determine what would be the diverse degrees of expectation that each of the Players would have to win a certain sum which would be the money of the game.

SOLUTION.

It is clear that each of the Players in order to wager equally & without disadvantage, must set into the game in ratio more or less by right that he has on the game, or the expectation that he has to win. One sees well, for example, that in case of primacy, Pierre has more advantage in this game than Paul, & Paul more advantage than Jacques, since it is able to be that Pierre wins without that Paul & Jacques having played, & also that Paul wins without that the turn of Jacques has come. But how much Pierre has more advantage than Paul, & Paul more advantage than Jacques, & what is, proportionally to these different advantages of the Players, the difference of the advances that each must make in order to compose the fund of the game? This is that which it is necessary to seek.

It is necessary to note first that the lot of a person who wagers to take a token among twelve, of which eight are blacks & four are whites, is to have one against two.

This supposed, if one names A the money of the game, S the lot of Jacques when Pierre is going to draw his token, y his lot when Paul is going to draw his, z his lot when it is to him to draw, one will have these three equalities $S = \frac{2}{3}y$, $y = \frac{2}{3}z$, $z = \frac{1}{3}A + \frac{2}{3}S$; whence one will draw $S = \frac{4}{19}A$, that which expresses the lot of Jacques.

Similarly in order to find the lot of Pierre, I name u his lot when he draws his token, t his lot when Paul draws his, q his lot when Jacques draws his token. Thus supposed, I have these three equalities $u = \frac{1}{3}A + \frac{2}{3}t$, $t = \frac{2}{3}q$, $q = \frac{2}{3}u$; whence one draws $u = \frac{9}{19}A$, that which expresses the lot of Pierre. Now the lot of Paul being to have the money of the game less the sum of the just claims of Pierre & of Jacques, one will have the lot of Paul $= A - \frac{4}{19}A - \frac{9}{19}A = \frac{6}{19}A$. Consequently if one wishes that the game be nineteen écus, it will be necessary that Pierre set nine, Paul set six, & Jacques four.

PROBLEM III.
PROPOSITION XXXIX.

Pierre wagers against Paul that taking, eyes closed, seven tokens among twelve, of which eight are blacks & four are whites, he will take from them three whites & four blacks. One demands how much Pierre & Paul must wager in order that the stake of each is in the same proportion as their lot.

SOLUTION.

It is necessary to seek first how many times eight tokens are able to be taken differently four by four, next how many times four tokens are able to be taken differently three by three; to multiply the number that gives the combination of eight tokens, taken four by four, by the number that gives the combination of four tokens taken three by three; this product will express all the coups that Pierre has in order to win. If one divides next this product by the number which expresses in how many different ways seven tokens are able to be taken in twelve, the exponent of this division will express the sought lot.

One will find by the Table, page 54, Proposition 10, that eight tokens are able to be taken differently seventy times four by four; that four tokens are able to be taken differently four times three by three, & finally that twelve tokens are able to be taken seven by seven in seven hundred ninety-two different ways. One will have therefore $\frac{70\times4}{792} = \frac{280}{792}$ for the expression of the lot of Pierre. Consequently the lot of Paul will be $\frac{512}{792}$. Therefore if one wishes that the fund of the game be one pistole, it will be necessary in order that the wager be equal, that Pierre sets three livres ten sols eight deniers, & Paul six livres nine sols four deniers.

PROBLEM IV.
PROPOSITION XL.

Pierre wagers against Paul that drawing, eyes closed, four cards among forty, namely ten diamonds, ten hearts, ten spades & ten clubs, he will draw one of each kind from them. One demands what is the lot of these two Players, or that which one must set into the game in order to wager equally.

SOLUTION.

One will find by the Table that has served to the solution of the preceding Problem, that forty cards are able to be taken four by four in ninety-one thousand three hundred ninety different ways. Now in this number which expresses all the possible ways by which forty cards are able to be taken differently four by four, there are ten thousand favorable to Pierre. In order to see it, it is necessary to remark that if there were only ten diamonds & ten hearts, there would be one hundred possible ways to take in these twenty cards of these two kinds; because each of the ten diamonds would be able to be taken with the ace of hearts, that which makes ten; or else with the deuce, that which makes again ten, & thus consecutively. Each of the ten diamonds would be able therefore to be taken with each of the ten hearts, that which makes one hundred. Presently if to these ten diamonds & to these ten hearts one adds ten clubs, it is clear that in order to have all the possible ways to take three cards of different kind among these thirty, it is necessary to multiply by ten the one hundred ways of which two cards of different kind are able to be taken among twenty, of which ten are diamonds & ten are hearts.

In order to comprehend more easily, one is able to imagine a card which has one hundred points in the place of the one hundred different ways, of which two cards of different kind are able to be taken in twenty cards, of which ten are diamonds & ten are hearts. Then one will note without difficulty that each of these one hundred points will be able to be found with the ace of clubs, that which makes one hundred; next each of these one hundred points with the deuce of clubs, that which will be two hundred; & with the three, that which will be three hundred; & finally the one hundred points successively with the ten clubs, that which will be one thousand. That being imagined, one will observe that easily that the fourth power of ten, which is ten thousand, expresses in how many ways one is able to take four cards of different kind among forty, which are ten diamonds, ten hearts, ten clubs & ten spades.

One will have therefore the lot of Pierre $= \frac{10000}{91390}A$, & consequently the one of Paul $= \frac{81390}{91390}A$.

COROLLARY.

If one demanded how much the odds be that Paul drawing thirteen cards at random in fifty-two, will not draw all one color, one would find that the odds are 158753389899 against 1.

If one demanded how much the odds be that Pierre drawing ten cards at random among forty, namely one ace, one deuce, one three, one four, one five, one six, one seven, one eight, one nine & one ten of diamonds, as many hearts, spades & clubs, he will draw a complete set of ten, one would find that there is odds 1048576 against 846611952, nearly 1 against 808.

PROBLEM V.
PROPOSITION XLI.

Pierre & Paul playing together at dice; Jacques, who is a third, hold twenty-four tokens in his hands, & will cast three dice as many coups as it will be necessary in order that both of the two Players win. Here are the conditions of the game. When the dice will bring forth eleven, Pierre will take one token among the twenty-four which are in the hands of Jacques; & when the dice will bring forth fourteen, Paul will take one of them; & the first of the two who will have taken twelve tokens will win. One demands what is the lot of the two Players.

SOLUTION.

It is necessary to note first that three dice give two hundred sixteen coups, since two dice give thirty-six coups, with each of which each face of the third die is found successively. One will observe next that among these two hundred sixteen coups there are twenty-seven which give a token to Pierre, namely six four & ace which arrives in six ways, six three & two which arrives in six ways, five four & two which arrives in six ways, five five & ace which arrives in six ways, four four & three which arrives in three ways, three three five which arrives in three ways: And fifteen which give a token to Paul, namely six five three which arrives in six ways, five five four which arrives in three ways, six six two which arrives in three ways, four four six which arrives in three ways.

This supposed, let

A be named the money of the game,
x the lot of Pierre when Pierre & Paul has nothing,
y the lot of Pierre when Pierre has one token & when Paul has nothing,
z the lot of Pierre when Pierre has two tokens & when Paul has nothing,
u the lot of Pierre when Pierre has three tokens & when Paul has nothing,
t the lot of Pierre when Pierre has four tokens & when Paul has nothing,

r the lot of Pierre when Pierre has five tokens & when Paul has nothing,
s the lot of Pierre when Pierre has six tokens & when Paul has nothing,
q the lot of Pierre when Pierre has seven tokens & when Paul has nothing,
p the lot of Pierre when Pierre has eight tokens & when Paul has nothing,
n the lot of Pierre when Pierre has nine tokens & when Paul has nothing,
m the lot of Pierre when Pierre has ten tokens & when Paul has nothing,
o the lot of Pierre when Pierre has eleven tokens & when Paul has nothing,
K the lot of Pierre when Pierre has nothing & when Paul has one token,
i the lot of Pierre when Pierre has nothing & when Paul has two tokens,
l the lot of Pierre when Pierre has nothing & when Paul has three tokens,
h the lot of Pierre when Pierre has nothing & when Paul has four tokens,
g the lot of Pierre when Pierre has nothing & when Paul has five tokens,
f the lot of Pierre when Pierre has nothing & when Paul has six tokens,
e the lot of Pierre when Pierre has nothing & when Paul has seven tokens,
d the lot of Pierre when Pierre has nothing & when Paul has eight tokens,
c the lot of Pierre when Pierre has nothing & when Paul has nine tokens,
b the lot of Pierre when Pierre has nothing & when Paul has ten tokens,
w the lot of Pierre when Pierre has nothing & when Paul has eleven tokens.

One will have

$$x = \frac{27y + 15K + 174x}{216}, \qquad y = \frac{27z + 15x + 174y}{216},$$
$$z = \frac{27u + 15y + 174z}{216}, \qquad u = \frac{27t + 15z + 174u}{216},$$
$$t = \frac{27r + 15u + 174t}{216}, \qquad r = \frac{27s + 15t + 174r}{216},$$
$$s = \frac{27q + 15r + 174s}{216}, \qquad q = \frac{27p + 15s + 174q}{216},$$
$$p = \frac{27n + 15q + 174p}{216}, \qquad n = \frac{27m + 15p + 174n}{216},$$
$$m = \frac{27o + 15n + 174m}{216}, \qquad \omega = \frac{27A + 15m + 174\omega}{216},$$
$$K = \frac{27x + 15i + 174K}{216}, \qquad i = \frac{27K + 15l + 174i}{216},$$
$$l = \frac{27i + 15h + 174l}{216}, \qquad h = \frac{27l + 15g + 174h}{216},$$
$$g = \frac{27h + 15f + 174g}{216}, \qquad f = \frac{27g + 15e + 174f}{216},$$
$$e = \frac{27f + 15d + 174e}{216}, \qquad d = \frac{27e + 15c + 174d}{216},$$
$$c = \frac{27d + 15b + 174c}{216}, \qquad b = \frac{27c + 15w + 174b}{216},$$
$$w = \frac{27b + 15 \times o + 174w}{216},$$

From all these equalities one will deduce $y = \frac{31381059609A + 39165289355x}{70546348964}$, & $K = \frac{70497520839}{70546348964}x$; & consequently one will have

$$x = \frac{27y + 15K + 174x}{216} = \frac{282429536481 \times A + 352487604195x + 352487604195 \times x}{987648885496},$$

by substituting for y & K their values in x, & finally one will have by reducing, $x = \frac{282429536481}{282673677106}A$, that which expresses the lot of Pierre; & $A - x = \frac{244140625}{282673677106}A$, that which expresses the lot of Paul.

PROBLEM VI.
PROPOSITION XLII.

To determine generally the divisions that one must make among many Players who play in an equal game in many parts.

Although this Problem is the least difficult of all those that one is able to be proposed on this matter, the conditions of the game being equal for all the Players, it has not abandoned exercising a long time, & to that which appears with pleasure, two illustrious Geometers, Messrs Fermat & Pascal. The latter employed in order to come to the end the analytic method; this way seems to be here the most natural & the easiest, but it has the defect of being excessively long, because one is not able to find the solution of the cases a little composed unless one has traversed all those which are less it, by commencing with the simplest. Thus, for example, in order to find by this way the lot of three Players Pierre, Paul & Jacques, by supposing that Pierre plays for one point, Paul for two, & Jacques for three, it would be necessary to examine what would be their lot, if Pierre playing for one point, Paul played similarly only for one point, & Jacques either for one, or for two, or for three points; (2) what would be their lot if Pierre playing for two points, Paul & Jacques played similarly for two points, that which would fall next into the previous case.

The method of Mr. Fermat is most scholarly, & demands more skill in its application. He has employed it only in order to determine the divisions among two Players. Mr. Pascal has not believed that it could be extended to a greater number. I will show that the method of Mr. Fermat resolves the Problem of division in a very general manner. But in order to make it understood, & to make known the difficulties that Mr. Pascal found, I believe to be able to do better only to report here his Letter of 24 August 1654 which is all on this subject. It is addressed to Mr. Fermat, & is found in his posthumous Works printed *in folio* at Toulouse: One will see the explication of the method of Mr. Fermat for two Players, & the doubts of Mr. Pascal on this method when one wishes to apply it to a greater number. I will give next the solution of the difficulties of Mr. Pascal, & I will apply this method to some Examples, which will make it known universally.

Letter from Mister Pascal to Mister de Fermat

Of 24 August 1645

SIR,

I could not offer to you my entire thought touching on the divisions for several Players in the last post, & likewise I have some repugnance to do it, for fear that by this, that admirable propriety, which was between us & which was so dear to me, begins to be refuted, for I believe that we are of different opinions on this subject. I wish to offer to you all my reasonings, & you will do me the favor to correct me if I err, or to affirm me, if I have met well. I ask this of you earnestly & sincerely, for I will hold myself for certain only when you will be on my side.

When there are only two Players, your method, which proceeds by combinations, is very sure; but when there are three I believe I have a demonstration that it is not correct, if it is only you proceed in some other manner that I do not understand; but the method that I have offered to you, & of which I serve myself throughout is common to all the conditions imaginable in all sorts of games, instead that of the combinations (which serves me only in the particular encounters where it is shorter than the general) is good only for those sole occasions & not in the others.

I am sure that I will give myself to understand, but a little discourse will be necessary for me & a little patience by you.

Here is how you precede when there are two Players:

If two Players, playing several games, find themselves in that state that two games are lacking to the first & three to the second, in order to find the division, it is necessary, say you, to see in how many games the game will be decided absolutely.

It is easy to suppose that this will be in four games, whence you conclude that it is necessary to see in how many ways four games are arranged between two Players & to see in how many ways there are combinations making the first win & in how many for the second, & to divide the money according to this proportion. I would have had difficulty to understand this discourse, if I had not known it by myself already; also you had written with this thought. Therefore, in order to see in how many ways four games are combined between two Players, it is necessary to imagine that they play with a die with two faces (since there are only two Players) as in heads & tails, & that they cast four of these dice (because they play to four games); & now one it is necessary to see in how many ways these dice have different states. This is easy to calculate; they are able to have sixteen which is the second degree of four, that is the square, for we figure that one of the faces is marked A, favorable to the first Player, & the other B favorable to the second; thus these four dice are able to be turned up on one of the following sixteen states.

a	a	a	a	1
a	a	a	b	1
a	a	b	a	1
a	a	b	b	1
a	b	a	a	1
a	b	a	b	1
a	b	b	a	1
a	b	b	b	2
b	a	a	a	1
b	a	a	b	1
b	a	b	a	1
b	a	b	b	2
b	b	a	a	1
b	b	a	b	2
b	b	b	a	2
b	b	b	b	2

And since the first Player lacks two games, all the faces which have $2A$ make him win; therefore there are 11 of them for him; & since the second lacks three games, all the faces where there are $3B$ are able to make him win; therefore there are 5 of them.

Therefore it is necessary that they divide the sum as 11 to 5. Here is your method when there are two Players. On which you say that, if there are more, it will not be difficult to find the division by the same method.

Upon this, Sir, I have to say to you that this division for two Players, founded on combinations, is very just & very good. But that if there are more than two Players, it will not always be just, & I will say to you the reason for this difference.

I communicated your method to our Colleagues, upon which Mr. de Roberval made this objection to me.

That which is wrong is that one takes the art to make the division under the supposition that one plays to four games, seeing that, when the one lacks two games & the other three, it is not of necessity that one plays four games, being able to happen that one will play only two or three, or in truth perhaps four.

And although he could not see why one would claim to make the just division under a make-believe condition that one will play four games, seeing that the natural condition of the game is that one will throw dice no longer if one of the Players will win, & that at least, if that were not false, that would not be demonstrated.

So that he had some suspicion that we had made a paralogism. I responded to him that I myself did not rely so much on the method of combinations, which truly is not in its place on this occasion, as on my other universal method, by which nothing escapes & which bears its demonstration by itself, which finds the same division precisely as that of the combinations; & moreover I will demonstrate to him the truth of the division between two Players by the combinations in this way.

Is it not true that, if two Players, finding themselves in the hypothetical state that one lacks two games & the other three, agree now by private contract that one plays four complete games, that is that one casts the four dice with two faces at the same time, is it not true, I say, that, if they have deliberated to play the four games, the division must be such as we have said, according to the multitude of states favorable to each.

He agreed with this & that indeed it is demonstrated, but he denied that the same thing subsisted in not being compelled to play the four games. I said to him therefore thus:

Is it not clear that the same Players, not being compelled to play the four games, but wishing to quit the game as soon as one had attained his number, is able with neither loss nor gain to be compelled to play the four entire games & that this convention changes in no manner their condition. For if the first wins the first two games out of four & that thus he has won, would he refuse to play yet two games, seeing that, if he wins them, he has not won more, & if he loses them, he has not won less; because these two that the other has won do not suffice for him, since three is necessary for him, & thus there is not enough in four games in order to make that both are able to attain the number which they lack.

Certainly it is easy to consider that it is absolutely equal & indifferent to the one & to the other to play to the natural condition in their game, which is to end as soon as one will have his count, or to play the entire four games: therefore since these two conditions are equal & indifferent, the division must be entirely equal to the one & to the other: now it is just when they are obliged to play four games, as I have shown.

Therefore it is just also in the other case. Here is how I demonstrated it, & if you take care, this demonstration is based on the equality of the two true & imagined conditions, in regard to two Players; & that in the one & in the other one same will always win, & if he wins or loses in the one, he will win or lose in the other, & never will both have their count. Let us follow the same point for three Players.

And let us suppose that the first lacks one game, that the second lacks two & the third two: in order to make the division according to the same method of combinations, it is necessary to seek first in how many games the game will be decided, as we have done when there were two Players: this will be in 3. for they would not know how to play three games unless the decision is necessarily arrived.

It is necessary to see now in how many ways three games are combined among three Players, & in how many ways they are favorable to the one, how many to the other, & how many to the last, & according to that proportion to distribute the money likewise as one has done under the hypothesis of two Players.

In order to see how many combinations there are in all, this is easy, it is the third power of 3, that is its cube 27.

For, if one throws three dice at the same time (since it is necessary to play three games), which each have three faces (since there are three Players), the one marked A favorable to the first, the other B for the second, the other C for the third.

It is clear that these three dice cast together are able to turn up in 27 different states, namely,

a	a	a	1			b	a	a	1			c	a	a	1		
a	a	b	1			b	a	b	1	2		c	a	b	1		
a	a	c	1			b	a	c	1			c	a	c	1		3
a	b	a	1			b	b	a	1	2		c	b	a	1		
a	b	b	1	2		b	b	b		2		c	b	b		2	
a	b	c	1			b	b	c		2		c	b	c			3
a	c	a	1			b	c	a	1			c	c	a	1		3
a	c	b	1			b	c	b		2		c	c	b			3
a	c	c	1		3	b	c	c			3	c	c	c			3

Now the first lacks only one game: therefore all the states where there is an A are for to him: therefore there are 19 of them.

The second lacks two games: therefore all the states where there are $2B$'s are for him: therefore there are 7 of them.

The third lacks two games: therefore all the states where there is $2C$'s are for him: therefore there are 7 of them.

If thence one concluded that it was necessary to give to each according to the proportion of 19, 7, 7, one would be deceived most grossly & I have maintained to believe that you would do thus. For there are some faces favorable to the first & to the second together, such as ABB, for the first Player finds one A which is necessary to him, & the second $2B$'s which are lacking to him; & thus ACC is for the first & and the third.

Therefore it is not necessary to count these faces which are common to two as being worth the entire sum to each, but only the half. For, if the state ACC happened, the first & the third would have the same right to the sum, having each their count, therefore they would divide the money in half; but if the state AAB arrives, the first wins alone. It is necessary to make the calculation thus:

There are 13 states which give the whole to the first & 6 which give the half to him & 8 which are worth nothing to him.

Therefore, if the entire sum is one pistole,

There are 13 faces which are each worth to him one pistole. There are 6 faces which are each worth to him $\frac{1}{2}$ pistole.

And 8 which are worth nothing.

Thus, in case of division, it is necessary to multiply

	13	by one pistole, which makes		13
	6	by one half, which makes		3
	8	by zero, which makes		0
Sum	27		Sum	16

And to divide the sum of the values 16 by the sum of the states 27 which makes the fraction $\frac{16}{27}$; which is that which belongs to the first in the case of division, namely 16 pistoles of 27.

The division for the second & for the third Player will be found likewise.

There are	4	states which are worth 1 pistole to him: multiply	4
There are	3	states which are worth $\frac{1}{2}$ pistole to him: multiply	$1\frac{1}{2}$
And	20	states which are worth nothing to him	0
Sum	27	Sum	$5\frac{1}{2}$

Therefore, five & $\frac{1}{2}$ pistoles out of twenty-seven belong to the second Player, & as much to the third, & these three sums $5\frac{1}{2}, 5\frac{1}{2}$ & 16 being joined, make twenty-seven.

Here is, it seems to me, in what manner it would be necessary to make the divisions by the combinations, according to your method, unless you have some other thing on this subject that I am unable to know.

But, if I am not deceived, this division is not just.

The reason for this is that one supposes a false thing, which is that one plays in three games infallibly, instead that the natural condition of this game here is that one plays only until one of the Players has attained the number of games which are lacking to him, in which case the game ceases.

It is not that it is able to happen that one of them plays three games, but it is able to happen also that one will play only one or two, & nothing of necessity.

But whence comes, one will say, that it is not permitted to make in this encounter the same make-believe supposition as when there were two Players?

Here is the reason for it.

Under the true condition of these three Players, there is only one who is able to win: for the condition is that, as soon as one has won, the game ceases; but, under the make-believe condition, two are able to attain the number of their games: namely if the first wins one of them which he lacks, & one of the others, two which they lack; for they would have played only three games, instead that, when there were only two Players, the make-believe condition & the true agreed for the advantages of the Players in all; & it is that which sets the extreme difference between the make-believe condition & the real.

But if the Players, finding themselves in the state of the hypothesis, that is, if the first lacks one game & the second two & the third two, wish now mutually & agree with this condition that one will play three complete games, & that those who will have attained the number lacking to them will take the entire sum, if it is found one alone who had attained it, or, if it is found that two had attained it, that they will divide it equally.

In this case the division must be made as I just gave it, that the first has 16, the second $5\frac{1}{2}$, the third $5\frac{1}{2}$ of twenty-seven pistoles, & that this bears its demonstration by itself by supposing this condition thus.

But if they play simply with the condition not that one necessarily plays three games, but only until this that one of among them has attained his games, & that then the game ceases without giving means to another to arrive there, then seventeen pistoles belong to the first, five to the second, five to the third of twenty-seven.

And this is found by my general method which determines also that under the preceding condition 16 of them is necessary to the first, $5\frac{1}{2}$ to the second, & $5\frac{1}{2}$ to the third without serving myself of combinations, for it goes especially alone & without obstacle.

Here are, Sir, my thoughts on this subject, about which I have no other advantage over you than the one of having much more mediation. But it is a small thing in your regard, since your first views are more penetrating than the length of my efforts.

I do not permit to offer to you my reasons for awaiting judgment from you.

I believe you to have made understood thence that the method of combinations is good between two Players by accident, as it is also sometimes between three Players, as when one lacks one game, one to the other & two to the other, because in this case the number of games in which the game will be achieved do not suffice in order to make two win; but it is not general & is good generally only in the case solely if one is compelled to play a certain number of games exactly.

So that as you did not have my method when you have proposed to me the division of many Players, but only that of combinations, I believe that you are of different sentiments on this subject; I beg you to send me in what way you proceed in the research of this division.

I will receive your response with respect & with joy, even when your sentiment will be contrary to mine. I am, &c.

PASCAL

The respect that we have for the reputation & for the memory of Mr. Pascal, does not permit us to remark here in detail all the faults of reasoning which are in this Letter; it will suffice for us to avert that the cause of his error is in not having regard to the diverse arrangements of letters.

In order to prove that of twenty-seven different situations that the three dice are able to have, there are seventeen which make Pierre win, & five which make each of the two other Players win to whom there is lacking two points; here is how it seems to me that one must reason.

The three Players oblige to play three games, but with this condition that if Pierre by which he lacks only one point, he wins before one or the other of the other Players has won two points, he will win the game; & that he will lose it if one or the other Player to whom there are lacking two points, is able to take them before Pierre has taken one of them. It is evident that this supposition reverts precisely to that of the Problem. Now according to this supposition one will find that of the twenty-seven situations of three dice there are seventeen which make Pierre win, five which make Paul win, & five which make Jacques win, thus it appears by the following Table.

TABLE

		Pierre		Paul	Jacques
aaa	*abc*	*bab*	*cac*	*bba*	*cca*
aab	*aca*	*bac*	*cba*	*bbb*	*ccc*
aac	*acb*	*bca*		*bbc*	*ccb*
aba	*acc*	*caa*		*bcb*	*cbc*
abb	*baa*	*cab*		*cbb*	*bcc*

REMARK I.

The general rule, that is to examine in how many coups at most the game must necessarily end; to take so many dice as there are of these coups, & to give to these dice as many faces as there are Players; next the concern is no more but to determine among all the possible dispositions of these dice, what are those which are advantageous & contrary to each of the Players, that which one will find always easily by the Problem of Proposition 30. Thus, for example, by supposing that Pierre plays for one point, Paul for two, & Jacques for three, if one wishes namely the lot of each of these three Players, it will be necessary in order to uncover it to imagine four dice marked with three points each, for example, with one 1, with one 2 & with one 3; to seek next by our rules of the combinations in how many ways he is able to find one ace which precedes either two 2, or three 3, & in how many ways two 2 or three 3 are able to precede the ace, that which will give the following Table.

TABLE

	Pierre.	*Paul.*	*Jacques.*
1, 1, 1, 1	1	0	0
1, 1, 1, 2	4	0	0
1, 1, 1, 3	4	0	0
1, 1, 2, 2	5	1	0
1, 1, 3, 3	6	0	0
1, 1, 2, 3	12	0	0
1, 2, 2, 3	8	4	0
1, 2, 3, 3	12	0	0
1, 2, 2, 2	3	0	1
1, 3, 3, 3	3	0	1
2, 2, 2, 2	0	1	0
2, 2, 2, 3	0	4	0
2, 2, 3, 3	0	6	0
2, 3, 3, 3	0	0	4
3, 3, 3, 3	0	0	1

Whence it appears that out of eighty-one coups there are fifty-seven for Pierre, eighteen for Paul, & six for Jacques.

One is able to resolve the preceding Problem in a briefer manner, by making the reasoning which follows.

I note that one would do wrong to none of these Players, if one obliged them to play three coups with these conditions. (1) That if Pierre won a coup before Paul had won two of them, he would be counted to have won the Game. (2) That if Paul won two coups before Pierre had won one, Paul would win. (3) That Jacques would have won if he won the three coups. (4) That if of the three coups Paul had won one of them, & Jacques two, the Players would separate themselves by retiring each his stake.

In order to calculate all this easily, one is able, as above, to imagine three dice which having each three faces, that on one is an ace, on the other a 2, on the third a 3, & to suppose that out of the twenty-seven coups that one is able to bring forth with these three dice, all those where there will be found an ace which precedes two 2 will be favorable to Pierre, & that all those where two 2 will precede the aces will be for Paul. One will find by Proposition 30 that there are eighteen coups which give A to Pierre, by supposing that A expresses all the money in the game, namely 1, 1, 1, which arrives in one way alone; 1, 1, 2; 1, 1, 3; 1, 3, 3, each in three ways; 1, 2, 3, which arrives in six ways; & these two here 1, 2, 2; 2, 1, 2. That there are five favorable to Paul, namely 2, 2, 1; 2, 2, 2, & 1, 2, 3 in three ways; & one coup alone which gives A to Jacques. One will find next that there are three coups which give $\frac{1}{3}A$ to each of the Players, namely 2, 3, 3,.

It is easy to note what are the cases where one is able to abbreviate in this manner the general method.

REMARK II.

When there are many Players to whom many points are lacking, the method which precedes by combinations & changes of order, is so long,& falls into such great detail than that which proceeds by Analysis, because one same coup of dice being able to be favorable to different Players, it appears that one is not able to dispense with considering that which each different coup of dice furnishes in particular, & this examination is able to be only quite long, even with the formula of Proposition 30. But the method of Mr. Fermat beyond many advantages that it has over that of Mr. Pascal, has the one to resolve in an infinitely short & simple manner the Problem in question, when the concern is only with two Players. Here is the rule.

Let Pierre be the one of two Players to whom is lacking the least points; let p be the number of points which is lacking to Pierre in order to win the game; q, the number of those which is lacking to Paul; let further $p + q - 1 = m$.

The lot of Pierre will be expressed by a fraction of which the denominator will be the number 2 raised to the exponent m, & of which the numerator will be composed of as many terms as this series, $1 + m + m \times \frac{m-1}{2} + m \times \frac{m-1}{2} \times \frac{m-2}{3} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c. as q expresses units. The lot of Paul will be the complement of unity.

EXAMPLES

Pierre plays for five points, & Paul for six points: the lot of Pierre is 319 against 193.

Pierre plays for four points, & Paul for six points: the lot of Pierre is 191 against 65, that which is a little less than 3 against 1.

Pierre plays for three points, & Paul for six points: the lot of Pierre is 219 against 37, that which is a little less that 6 against 1.

Pierre plays for two points, & Paul for six points: the lot of Pierre is 15 against 1.

Pierre plays for one point, & Paul for six points: the lot of Pierre is 63 against 1.

REMARK III.

If one played in a certain number of games always by reducing, that is, so that Pierre having for example three games of six, & coming to lose one of them, Paul marked nothing, & unmarked only one game to Pierre, & thus of the rest; the advantages of Pierre would be different & much less than one did not find them under the preceding supposition.

Here is a Table that I amused myself by making some time ago seeing a game of Piquet rather extraordinary. It was to the great one hundred, in order to win it was necessary to have six games, & one played always by reducing in the manner that we just explained. It seems that this condition must render the game interminable. It will happen in fact that the Players after having played thirty or forty games, one or the other approaching from time to time to the end without having been able to attain it, prays to renounce the game, & to be separated without finishing it: but as one of the Players had three points or three games marked before him, one agrees that he was just to divide the money of the game (it was eight louis, which made then 128 liv.) more equally than he would be able. I will find that the Player who had three tokens must withdraw precisely 96 liv. & the other Player 32 liv.

Under the assumption of the preceding Remark he would need to withdraw 109 liv. 10 sols, & the other 18 liv. 10 sols, because these two numbers are between them as 219 & 37, & their sum is 128.

TABLE.

Pierre has one point, his lot is 7 against 5.
Pierre has two points, his lot is 2 against 1.
Pierre has three points, his lot is 3 against 1.
Pierre has four points, his lot is 5 against 1.
Pierre has five points, his lot is 11 against 1.

PROBLEM
PROPOSITION XLIII.

Pierre playing against Paul has lost a certain sum of money, & having in order to pay only the half of this sum, he promises to acquit to him the other half in a certain number of equal payments. Paul consents, on condition that Pierre in these payments will comprehend a certain interest on which they agree. One demands how much will be these payments.

Let y be the payment that Pierre must make all the years to Paul, q the number which expresses on what sum Pierre takes one unit of interest per year, A the sum which remains to pay; let also, for brevity, $p = \frac{q}{q+1}$.

One will have $y = \frac{A}{p+pp+p^3+p^4+p^5+\&c.}$

It is necessary to note that the denominator will be composed of as many terms as Pierre will have taken years in order to achieve to pay his debt.

Example. If one supposes that Pierre owes ten thousand francs to Paul, & if he is obliged from them to pay to him in four equal payments from year to year, by comprehending the interest of five percent; one will find, by substituting into this formula for A & p their values, that each payment must be 2820 liv. 2 f. 4 d. $\frac{13732}{34481}$.

This Problem, which is quite easy for the Geometers, would be apparently quite awkward for the Arithmeticians, & as it is of a rather great usage, I have judged proper to set it here. I have sought to the occasion of a question which was given to me some time ago by one of my friends. Here it is.

He wished to buy a Land of which the principal revenue was in wood, there was one hundred acres in regulated cuts, & they were cut all in seventeen years. He had old oak among the cuttings. A Merchant proposed to him to give to him eighty livres of silver during the seventeen years by cutting the oak with the cuttings. Moreover, the Vendor was certain by the estimation of the Connoisseurs, that the great oaks being cut, the cuttings, which at the end of seventeen years would be nicer income, would be worth forty livres of silver. On this exposed he will demand of me how much he would be able to evaluate the property of these woods, by estimating this acquisition on the basis of the last 20. I found by the preceding formula, that it would be evaluated, according to its just value, at more than one hundred twenty-five thousand livres, & at least one hundred twenty-five thousand one hundred livres.

PROBLEM
PROPOSITION XLIV.

The number which expresses the ratio of the lot of Pierre to the one of Paul, in supposing that Pierre wagers against Paul to make a certain thing on the first coup, being given, One demands what is the number which expresses the lot of Pierre, in supposing that one accords to him a certain number of coups in order to make the proposed thing.

If one expresses by the unknown x his lot when having lacked to win on the first coup he goes to replay his second coup, & y his lot when having lacked to win on the second coup he goes to replay his third coup, & z his lot when having lacked to win at his third coup he goes to replay a fourth of them, &c. employing in sequence the letters x, y, z, u, t, r, &c. in order to express the unknown lot of Pierre at his second, third, fourth, fifth, sixth, seventh coup, &c. Naming again p the number of encounters favorable to Pierre, q the number of encounters favorable to Paul, & supposing $m = p + q$, one will have the lot of Pierre at the commencement of the game $= \frac{p}{m} + \frac{q}{m}x$, $x = \frac{p}{m} + \frac{q}{m}y$, $y = \frac{p}{m} + \frac{q}{m}z$, $z = \frac{p}{m} + \frac{q}{m}u$, $u = \frac{p}{m} + \frac{q}{m}t$, $t = \frac{p}{m} + \frac{q}{m}r$, &c. & substituting all these quantities, one will have the lot of Pierre expressed by this infinite series $S = \frac{p}{m} + \frac{pq}{mm} + \frac{pqq}{m^3} + \frac{pq^3}{m^4} + \frac{pq^4}{m^5} + \frac{pq^5}{m^6} + \frac{pq^6}{m^7} +$ &c. & adding as many terms of this series as it will be necessary in order that S is equal to $\frac{1}{2}$. One will conclude that Pierre is able to undertake the wager end to end as many coups as one will have employed terms of this series in order to have $S = \frac{1}{2}$, or a little greater.

This method is quite simple, & is scarcely different from that Mr. Huygens employed in order to determine in how many coups one is able to wager end to end to bring forth sonnez with two dice; but they have both this inconvenience that they are absolutely impractical when p being a small number, m & q express great of them. For example if one sought in how many coups Pierre would be able to wager with some advantage to have carte blanche in Piquet, it is clear that it would be necessary to add more than one thousand terms of this series, in which p being 323, q would be 578633, & m 578956, & that a work of many years sufficed with difficulty for this annoying computation.

Here is the way to avoid a calculation of such great length.

I observe, (1) that the sum of this infinite series is always equal to unity, since it is clear that if there is some possibility that Pierre wins on the first coup, there is certitude that he will win having an infinite number of coups to play in sequence. I substitute therefore this fraction $\frac{p}{m-q}$ in the place of unity, & I make the division in the numeric manner, there comes to me for quotient the fraction $\frac{p}{m} + \frac{p \times q}{mm} + \frac{p \times qq}{m^3} + \frac{p \times q^3}{m^4} +$ &c.. I note in second place that by making this division there is always a remainder which for the first operation is $+\frac{pq}{m}$, for the second $+\frac{pqq}{mm}$, for the third $\frac{pq^3}{m^3}$, for the fourth $\frac{pq^4}{m^4}$, so that $\frac{p}{m} = \frac{p}{m-q} - \frac{\frac{pq}{m}}{m-q}$, & $\frac{p}{m} + \frac{pq}{mm} = \frac{p}{m-q} - \frac{\frac{pqq}{mm}}{m-q}$, & $\frac{p}{m} + \frac{pq}{mm} + \frac{pqq}{m^3} = \frac{p}{m-q} - \frac{\frac{pq^3}{m^3}}{m-q}$, & $\frac{p}{m} + \frac{pq}{mm} + \frac{pqq}{m^3} + \frac{pq^3}{m^4} = \frac{p}{m-q} - \frac{\frac{pq^4}{m^4}}{m-q}$; whence it is clear that the number of terms of the series that one wishes to add into one sum being h, one will have all these terms together equal to $\frac{p}{m-q} - \frac{\frac{pq^h}{m^h}}{m-q}$.

Thence I draw this rule, that in order to find the number of coups which would render the lot of Pierre equal to the one of Paul, it is necessary to subtract from unity the quantity $\frac{\frac{pq^h}{m^h}}{m-q}$, in which h has such a value that this fraction $\frac{\frac{pq^h}{m^h}}{m-q}$ is $\frac{1}{2}$ or less than $\frac{1}{2}$; & consequently it is necessary that $m^{h+1} - q \times m^h$ be greater than $2 \times m \times q^h - q^{h+1}$; or, dividing all by $m - q$, m^h greater than $2 \times q^h$. That which will serve as formula.

FIRST EXAMPLE.

Let be supposed that one seeks in how many coups Pierre is able to wager to bring forth six with one die, it will be necessary to substitute 6, 5, 1 for the letters m, q, p, so that one will have $1 - \frac{5^h}{6^h} = \frac{p}{m-q} - \frac{\frac{pq^h}{m^h}}{m-q}$, & one will know without difficulty that h being 4, that is, Pierre is proposing to bring forth six in four coups, there will be advantage for him, because $1 - \frac{5^4}{6^4} = 1 - \frac{625}{1296}$; now this fraction $\frac{625}{1296}$ is smaller than $\frac{1}{2}$ of the quantity $\frac{23}{1296}$ which will express the advantage of Pierre in wagering to bring forth a six with a die in four coups. One will know also that by substituting 3 for h, that is, that Pierre being proposed to bring forth a six in three coups, there would be for him disadvantage, & his disadvantage would be $\frac{17}{256}$.

It is clear that the exponent h must be so much greater as q is greater with respect to p, so that, for example, q being $= 5$, & consequently $m = 6$, h must be $= 4$, & q being $= 35$, h must be $= 25$.

SECOND EXAMPLE.

Let the lot of Pierre be $\frac{1000}{9139}$, one will have $m = 9139$, & $q = 8139$; let be supposed $h = 6$, one will find the logarithm of the number $9139 = 3.9608987$, that which being multiplied by 6 gives 23.7653922 logarithm of the number 9139 raised to the sixth power, & the logarithm of the number $8139 = 3.9105710$, that which being multiplied by 6 gives 23.4634260 logarithm of he number 8139 raised to the sixth power.

If one seeks the numbers which correspond to these logarithms, one will find $m^6 = 582628954909994978159161$ greater than $2 \times q^6 = 581374911690872909930322$.

Thus Pierre wagering against Paul to draw at random four cards of different color in a Game composed of 40 cards (see Proposition 40) his lot if he undertakes it in six coups will be to the one of Paul as

$$291941499064558523194000,$$

is to

$$290687455845436454965161,$$

& if he undertakes in five coups, as

$$28036559991735205000$$

is to

$$35715377300090484699.$$

REMARK.

In order to avoid groping, it will be necessary to convert the formula $m^h = 2 \times q^h$ into another where h is only in one of the members of the equality, that is is able by employing the calculus of exponentials. Because one finds that it is changed into this other $h = \frac{\log 2}{\log m - \log q}$, & this formula where h expresses the number of coups that one seeks, will give first the solution of the proposed Problem.

For example, if one wishes to know in how many ways one is able to wager to end to bring forth sonnez with two dice, one will find by substituting for m, 36, & for q, 35, $h = \frac{3010300}{15563025 - 15440680} = 24 + \frac{14804}{14469}$, that which shows that one would undertake with advantage in 25 coups, & with disadvantage in 24 coups.

And likewise if one seeks in how many coups one is able to wager to end to have carte blanche at Piquet (see page 68) one will have, by substituting into the formula for m, 578956, & for q, 578633, $h = \frac{3010300}{57626456 - 57624032} = 1241\frac{529}{606}$, & that which shows that one would undertake it with advantage in 1242 coups, & with disadvantage in 1241 coups.

One will be able likewise to discover by this way how much the advantage or disadvantage will be with respect to such number of coups as it be.

COROLLARY.

One will be able, by the method of this Problem, to resolve the one which follows: *To determine how much a game must endure where one would always play by reducing, according to the conditions that one has explicated, p. 111.* One will find, for example, that if one plays according to the rules in three games of Piquet, there is advantage to wager that the game will be ended in seven games, as of 37 to 27; & disadvantage to wager that it will be ended in five, as of 7 to 9. One will find in this formula $\frac{1}{4} + \frac{3^1}{4^2} + \frac{3^2}{4^3} + \frac{3^3}{4^4} + \frac{3^4}{4^5} +$&c. the ratio of the advantage or of the disadvantage that there is to wager that the game will be decided in a certain number of games whatsoever. The first term of this series expresses the lot of the one who would wager that the game will be decided in three coups; the sum of the first two expresses the lot of a Player who would wager that the game would be decided in five coups; the sum of the first three expresses the lot of a Player who would wager that the game would be decided in seven coups, &c. One will find without much difficulty some similar formulas for the other cases, & one will find the research of it rather curious.

PROBLEMS TO RESOLVE.
FIRST PROBLEM.
ON THE GAME OF TREIZE.

To determine generally what is in this game the advantage of the one who holds the cards. One will find the explication of the rules of this Game on page 36.

SECOND PROBLEM.
ON THE GAME CALLED HER.

One draws first the places, one sees next who will have the hand. Let us suppose that it is Pierre, & let us name the other Players Paul, Jacques & Jean.

One agrees to set a certain sum into the game; each of the Players take for this sum an equal number of tokens; & the one there wins all the money of the game which remains with one or many tokens, the other Players having no more. Here is how the game is conducted.

Pierre holds an entire deck composed of fifty-two cards, & gives one of them to each of the Players, by commencing at his right; & at the end of each coup the one who is found to have the lowest card, loses a token that he set in the middle of the table.

Paul who is the first to the right of Pierre, has right if he is not content with his card, to exchange with Jacques, who is able to refuse it to him only in the sole case that he has a King, then Jacques says *coucou*. With this term the one who has a King warned the Players that his neighbor to the left having wished to be undone of his card has been stopped by his. It is likewise of Jacques in regard to Jean, & of Jean in regard to Pierre.

It is necessary only to note, (1), that if Pierre is not content with his card, either that it is that which he is given first or that which he has been constrained to receive from Jean, he is able, having no person with whom to exchange, to hold to take a better card by cutting at random among those which remain to him in the hand. (2) That if it happens that Pierre having for example a five, does not wish to be held, & that cutting them he draws for example a Jack, his card will become a Jack, & thus of each other card, with the exception of the King, because Pierre drawing a King is sent away to his card such as it be, & he is found as if he was first held to his card.

All this change of cards being made, each Player uncovers his, & the one who is found to have the lowest commencing with the ace, sets a token into the game.

If it is encountered that two or many Players have the same card, & if this is the lowest, the one who has the primacy, that is the one who is the nearest to the right of Pierre loses & pays. That which shows that one must always hold when one has given to the Player who is to the left a card similar to that which one receives from him, likewise that if one had given to him one lower.

The Player who has lost his tokens exits from the order, & the others continue the game until all, with the exception of one alone, have lost all their tokens, in which case the one who remains wins the money of all the Players, & that is called in terms of Players *to win the pool*.

Here is the Problem of which one demands the solution.

Four Players, Pierre, Paul, Jacques & Jean are the only Players who remain, & they have no more than one token each. Pierre holds the cards, Paul is to his right, & the others next. One demands what is their lot with respect to the different place that they occupy, & with what proportion must divide the money of the pool, it will be for example ten pistoles, if they wished to divide it among them without finishing the game.

THIRD PROBLEM
ON THE GAME OF THE FARM.

One sets the Farm at price, & one awards to the one who carries the highest; for example if the tokens are worth twenty sols, one will carry it to two or three pistoles, & the Farmer will set them on the table. Here is the rules of this Game.

Each of the Players sets a token into the game, next the Farmer distributes two cards to them, namely one of above, & the second below. Those among the Players of whom the two cards are more than sixteen, give as many tokens to the Farmer as the cards are points above sixteen. For example if Paul who is one of the Players received first a nine, & for the second card a ten, that makes 19, he will pay three tokens to the Farmer. It is necessary to observe that in this game the ace is worth only one.

The Players of whom the two cards are less than sixteen have the liberty to be held under the fear of passing sixteen & to pay to the Farmer for the surplus. They have also the liberty to demand some new cards under the expectation either to win the Farm & the turns if they are able to attain precisely the number sixteen, or at least to approach below more nearly than any other Player, in which case they will win the turns.

When all the Players pass the number of sixteen, the turns remain in the game, & each Player sets anew a token.

In this game the number of the Players is undetermined.

One plays with an entire deck of cards, & sometimes one omits the six in order to prevent that the number sixteen is encountered too often.

When two or many Players have an equal number of points, the one who is the most to the right of the Farmer is the only one who wins. Thus the Farmer is never able to win the turn except when he had a point closer to sixteen than each other Player, & if he has sixteen at the same time as another Player, he would not permit to lose the Farm & one would make a new auction. All this being explicated, here is the Problem of which one demands the solution.

Being supposed a certain determined number of Players, for example two Players Pierre & Paul, & that the price of the tokens is twenty sols: One demands how much must be the price of the Farm, so that one was able to hold it with neither profit nor disadvantage.

FOURTH PROBLEM
ON THE GAME OF THE HEAP.

In order to comprehend of what there is concern, it is necessary to know that after the resumption of hombre one of the Players amuses himself often by dividing the deck into ten heaps each composed of four covered cards & that next turning over the first of each heap, he takes off & sets apart two by two all those which are found similar, for example two Kings, two Jacks, two ten, &c. then he turns over the cards which follow immediately those which come from them to give the doublets, & he continues to take off & to set apart those which come by doublet until he is come to the last of each heap, after having raised them all two by two, in which case only he has won.

It is rare that one plays with money in this game, but one plays often with prudence, & the Ladies please themselves to judge the event of certain triflings which interest them, by the success that they have in this game. It is necessary to observe that this game is not of pure chance, & that in order to succeed it is necessary to conduct it as well as from fortune.

One knows that it is necessary to unload the largest heap preferably to the small, but one does not know exactly if it is most advantageous to unload two heaps composed of three cards each, or two heaps of which one will be composed of four cards, & the other of two. One knows also that it is easier to make the heap with one deck of Piquet than with a deck of Hombre, & with a deck of Hombre than with an entire deck, or else with two games of Hombre you mix together, that which would make the heap of eight cards. But that which the Players ignore entirely, is the degree of facility that there is to be successful in all these different kinds. One demands a general method in order to *determine what is the advantage or the disadvantage of the one who undertakes to make the heap, either that it is with with one deck of Piquet, or with a deck of Hombre, or with an entire deck; & what is the manner to conduct his game most advantageous as it is possible.*

END.

APPROBATION.

I have read by order of Monseignor the Chancelor a manuscript entitled, *Essai d'Analyse sur les Jeux de hazard*. This work has appeared to me worthy of the curiosity of the most profound Analysts, & of a more considerable & more extended utility that seems not to promote the the subject which is treated. Made at Paris 24 February 1707. *Signed*, SAURIN.