FROM
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From the Introduction pp. 16–28

CASE VIIIth

If A and B play together, and that A wants but 1 Game of being up, and B wants 2; what are their respective Probabilities of winning the Set?

SOLUTION.

It is to be considered that the Set will necessarily be ended in two Games at most, for if A wins the first Game, there is no need of any farther Trial; but if B wins it, then they will want each but 1 Game of being up, and therefore the Set will be determined by the second Game: from whence it is plain that A wants only to win onece in two Games, but that B wants to win twice altogether. Now supposing that A and B have an equal Chance to win a Game, then the Probability which B has of winning the first Game will be $\frac{1}{2}$,and consequently the Probability of his winning twice together will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; and therefore the Probability which A has of winning once in two Games will be $1 - \frac{1}{4} = \frac{3}{4}$, from whence it follows that the Odds of A's winning are 3 to 1.

CASE IXth

A and B play together, A wants 1 Game of being up, and B wants 2; but the Chances whereby B may win a Game, are double to the number of Chances whereby A may win the same: 'tis required to assign the respective Probabilities of winning.

SOLUTION.

It is plain that in this, as well as in the preceding case, B ought to win twice altogether; now since B has 2 Chances to win a Game, and A 1 Chance only for the same, the Probability which B has of winning a Game is $\frac{2}{3}$, and therefore the Probability of his winning twice altogether is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and consequently the Probability of A's winning the Set is $1 - \frac{4}{9} = \frac{5}{9}$; from whence it follows that the Odds of A's winning once, before B twice, are as 5 to 4.

REMARK.

Altho' the determining the precise Odds in questions of Chance requires calculation, yet sometimes by a superficial View of the question, it may be possible to find that there will be an inequality in the Play. Thus in the preceding case wherein B has in every Gmae twice the number of Chances of A, if it be demanded whether A and B play upon the square, it is natural to consider that he who has a double number of Chances will at long run win twice as often as his Adversary; but that the case is here otherwise, for B undertaking to win twice before A once, he thereby undertakes to win ofner than according to his proportion of Chances, since A has a right to expect to win once, and therefore it may be concluded

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that B has the disadvantage: however, this way of arguing in general ought to be used with the utmost caution.

12. Whatever be the number of Games which A and B respectively want of being up, the Set will be concluded at the most in so many Games wanting one, as is the sum of the Games wanted between them.

Thus suppose that A wants 3 Games of being up, and B 5; it is plain that the greatest number of Games that A can win of B before the determination of the Play will be 2, and that the greatest number which B can win of A before the determination of the Play will be 4; and therefore the greatest number of Games that can be played between them before the determination of the Play will be 6: but supposing they have played six Games, the next Game will terminate the Play; and therefore the utmost number of Games that can be played between them will be 7, that is one Game less than the Sum of the Games wanted between them.

CASE X$^{\text{th}}$

Supposing that A wants 3 Games of being up, and B wants 7; but that the Chances which A and B respectively have for winning a Game are as 3 to 5, to find the respective Probabilities of winning the Set.

SOLUTION.

By reason that the Sum of the Games wanted between A and B is 10, it is plain by the preceding Paragraph that the Set will be concluded in 9 Games at most, and that therefore A undertakes out of 9 Games to win 3, and B, out of the same number, to win 7; now supposing that the first general Theorem[1] laid down in the 11$^{\text{th}}$ Art is particularly adapted to represent the Probability of A$^{\text{'s}}$ winning, then $l = 3$; and because n represents the number of Games in which the Set will be concluded, $n = 9$; but the number of terms to be used in the first Theorem[2] being $= n - l + 1 = 7$, and the number of terms to be used in the second Theorem being $= l = 3$, it will be more convenient to use the second, which will represent the Probability of B$^{\text{'s}}$ winning. Now that second Theorem being applied to the case of n being $= 9$, $l = 3$, $a = 3$, $b = 5$, the Probability which B has of winning the Set will be expressed by $\frac{5^7}{8^7} \times \left(1 + \frac{21}{8} + \frac{252}{64}\right) = \frac{5^7}{8^9} \times 484 = 0.28172$ nearly; and therefore subtracting this from Unity, there will remain the Probability which A has of winning the same, which will be 0.71828: and consequently the Odds of A$^{\text{'s}}$ winning the Set will be 71828 to 28172, or very near as 23 to 9.

The same Principles explained in a different and more general way.

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[1]Theorem 1. Let a be the number of Chances, whereby an Event may happen, b the number of Chances whereby it may fail, l the number of times that the Event is required to be produced in any given number of Trials, and let n be the number of those Trials; make $a + b = s$, then the Probability of the Event's happening l times in n Trials, will be expressed by the Series

$$\frac{a^l}{s^l} \times \left(1 + \frac{lb}{s} + \frac{l.l+1.bb}{1.2.ss} + \frac{l.l+1.l+2.b^3}{1.2.3.s^3} + \frac{l.l+1.l+2.l+3.b^4}{1.2.3.4.s^4} \text{ \&c.}\right)$$

which Series is to be continued to so many terms exclusive of the common multiplicator $\frac{a^l}{s^l}$ as are denoted by the number $n - l + 1$.

[2]Theorem 2. And for the same reason, the Probability of the contrary, that is of the Event's not happening so often as l times, making $n - l + 1 = p$, will be expressed by the Series

$$\frac{b^p}{s^p} \times \left(1 + \frac{pa}{s} + \frac{p.p+1.aa}{1.2.ss} + \frac{p.p+1.p+2.a^3}{1.2.3.s^3} + \frac{p.p+1.p+2.p+3.a^4}{1.2.3.4.s^4},\right)$$

which Series is to be continued to so many terms exclusive of the common multiplicator, as are denoted by the number l.

Altho' the principles hitherto explained are a sufficient introduction to what is to be said afterwards; yet it will not be improper to resume some of the preceding Articles, and to set them in a new light: it frequently happening that some truths, when represented to the mind under a particular Idea, may be more easily apprehended than when represented under another.

13. Let us therefore imagine a Die of a given number of equal faces, let us likewise imagine another Die of the same or any other number of equal faces; this being supposed, I say that the number of all the variations which the two Dice can undergo will be obtained by multiplying the number of faces of the one, by the number of faces of the other.

In order to prove this, and the better to fix the imagination, let us take a particular case: Suppose therefore that the first Die contains 8 faces, and the second 12; then supposing the first Die to stand still upon one of its faces, it is plain that in the mean time the second die may revolve upon its 12 faces; for which reason, there will be upon that single score 12 variations: let us now suppose that the first Die stands upon another of its faces, then whilst that Die stands still, the second Die may revolve again upon its 12 faces; and so on, till the faces of the first Die have undergone all their changes: from whence it follows, that in the two Dice, there will be as many times 12 Chances as there are faces in the first Die; but the number if faces if the first Die has been supposed 8, wherefore the number of Variations or Chances of the two Dice will be 8 times 12, that is 96: and therefore it may be universally concluded, that the number of all the variations of two Dice will be the product of the multiplication of the number of faces of one Die, by the number of faces of the other.

14. Let us now imagine that the faces of each Die are distinguished into white and black, that the number of white faces upon the first is A, and the number of black faces is B, and also that the number of white faces upon the second is a, and the number of black faces is b; hence it will follow by the preceding Article, that multiplying $A + B$ by $a + b$, the product $Aa + Ab + Ba + Bb$, will exhibit all the Variations of the two Dice: Now let us see what each of these four parts separately taken will represent.

1°. It is plain, that in the same manner as the product of the multiplication of the whole number of faces of the first Die, by the whole number of faces of the second, expresses all the variations of the two Dice; so likewise the multiplication of the number of the white faces of the first Die, by the number of the white faces of the second, will express the number of variations whereby the two Dice may exhibit two white faces: and therefore, that number of Chances will be represented by Aa.

2°. For the same reason, the multiplication of the number of white faces upon the first Die, by the number of black faces upon the second, will represent the number of all the Chances whereby a white face of the first may be joined with a black face of the second; which number of Chances will therefore be represented by Ab.

3°. The multiplication of the number of white faces upon the second, by the number of black faces upon the first, will express the number of all the Chances whereby a white face of the second may be joined with a black face of the first; which number of Chances will therefore be represented by aB.

4°. The multiplication of the number of black faces upon the first, by the number of black faces upon the second, will express the number of all the Chances whereby a black face of the first may be joined with a black face of the second; which number of Chances will therefore be represented by Bb.

And therefore we have explained the proper signification and use of the several parts Aa, Ab, Ba, Bb singly taken.

But as these parts may be connected together several ways, so the Sum of two or more of any of them will answer some question of Chance: for instance, suppose it be demanded, what is the number of Chances, with the two Dice above-mentioned, for throwing a white face? it is plain that the three parts $Aa + Ab + Ba$ will answer the question; since every one of those parts comprehends a case wherein a white face is concerned.

It may perhaps be thought that the first term Aa is superfluous, it denoting the number of Variations whereby two white faces can be thrown; but it will be easy to be satisfied of the necessity of taking it in: for supposing a wager depending on the throwing of a white face, he who throws for it, is reputed a winner, whenever a white face appears, whether one alone, or two together, unless it be expressly stipulated that in a case he throw two, he is to lose his wager; in which case the two terms $Ab + Ba$ would represent all his Chances.

If now we imagine a third Die having upon it a certain number of white faces represented by α, and likewise a certain number of black faces represented by β, then multiplying the whole variation of Chances of the two preceding Dice *viz.* $Aa + Ab + Ba + Bb$ by the whole number of faces $\alpha + \beta$ of the third Die, the product $Aa\alpha + Ab\alpha + Ba\alpha + Bb\alpha + Aa\beta + Ab\beta + Ba\beta + Bb\beta$ will exhibit the number of all the Variations which the three Dice can undergo.

Examining the several parts of this new product, we may easily perceive that the first term $Aa\alpha$ represents the number of Chances for throwing three white faces, that the second term $Ab\alpha$ represents the number of chances whereby both the first and the third Die may exhibit a white face, and the second Die a black one; and that the rest of the terms have each their particular properties, which are discovered by bare inspection.

It may also be perceived, that by joining together two or more of those terms, some question of Chance will thereby be answered: for instance, if it be demanded what is the number of Chances for throwing two white faces and a black one? it is plain that the three terms $Ab\alpha$, $Ba\alpha$, $Aa\beta$ taken together will exhibit the number of Chances required, since in every one of them there is the expression of two white faces and a black one; and therefore if there be a wager depending on the throwing two white faces and a black one, he who undertakes that two white faces and a black one shall come up, has for him the Odds of $Ab\alpha + Ba\alpha + Aa\beta$ to $Aa\alpha + Bb\alpha + Ab\beta + Ba\beta + Bb\beta$; that is, of the three terms that include the condition of the wager, to the five terms that include it not.

When the number of Chances that was required has been found, then making the number the Numerator of the fraction, whereof the Denominator is the whole number of variations which all the Dice can undergo, that fraction will express the Probability of the Event; as has been shewn in the first Article.

Thus if it was demanded what the Probability is, of throwing three white faces with the three Dice above-mentioned, that Probability will be expressed by the fraction

$$\frac{Aa\alpha}{Aa\alpha + Ab\alpha + Ba\alpha + Aa\beta + Bb\alpha + Ab\beta + Ba\beta + Bb\beta}.$$

But it is to be observed, that in writing the Denominator, it will be convenient to express it by way of product, distinguishing the several multiplicators whereof it is compounded; thus in the preceding case the Probability required will be best expressed as follows,

$$\frac{Aa\alpha}{(A+B) \times (a+b) \times (\alpha+\beta)}.$$

If the preceding fraction be conceived as the product of the three fractions $\frac{A}{A+B} \times \frac{a}{a+b} \times \frac{\alpha}{\alpha+\beta}$, whereof the first expresses the Probability of throwing a white face with the first Die; the second the Probability of throwing a white face with the second Die, and the third the

Probability of throwing a white face with the third Die; then will again appear the truth of what has been demonstrated in the 8th *Art.* and its *Corollary*, *viz.* that the Probability of the happening of several Events independent, is the product of all the particular Probabilities whereby each particular Event may be produced; for altho' the case here described be confined to three Events, it is plain that the Rule extends itself to any number of them.

Let us resume the case of two Dice, wherein we did suppose that the number of white faces upon one Die was expressed by A, and the number of black faces by B, and also that the number of white faces upon the other was expressed by a, and the number of black faces by b, which gave us all the variations $Aa+Ab+aB+Bb$; and let us imagine that the number of the white and black faces is respectively the same upon both Dice: wherefore $A=a$, and $B=b$, and consequently instead of $Aa+Ab+aB+Bb$, we shall have $aa+ab+ab+bb$, or $aa+2ab+bb$; but in the first case $Ab+aB$ did express the number of variations whereby a white face and a black one might be thrown, and therefore $2ab$ which is now substituted in the room of $Ab+aB$ does express the number of varieties, whereby with two Dice of ths same respective number of white and black faces, a white face and a black one may be thrown.

In the same manner, if we resume the general case of three Dice, and examine the number of variations whereby two white faces and a black one may be thrown, it will easily be perceived that if the number of white and black faces upon each Die are respectively the same, then the three parts $Ab\alpha+Ba\alpha+Aa\beta$ will be changed into $aba+baa+aab$, or $3aab$, and that therefore $3aab$, which is one term of the Binomial $a+b$ raised to its Cube, will express the number of variations whereby three Dice of the same kind would exhibit two white faces and a black one.

15. From the preceding considerations, this general Rule may be laid down, *viz.* that if there be any number of Dice of the same kind, all distinguished into white and black faces, that n be the number of those Dice, a and b the respective numbers of white and black faces upon each Die, and that the Binomial $a+b$ be raised to the power n; then 1°, the first term of that power will express the number of Chances whereby n white faces may be thrown; 2°, that the second term will express the number of Chances whereby $n-1$ white faces and 1 black face may be thrown; 3°, that the third term will express the number of Chances whereby $n-2$ white faces and 2 black ones may be thrown; and so on for the rest of the terms.

Thus, for instance, if the Binomial $a+b$ be raised to its 6th power, which is $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$; the first term a^6 will express the number of Chances for throwing 6 white faces; the second term $6a^5b$ will express the number of Chances for throwing 5 white and 1 black; the third term $a5a^4b^2$ will express the number of Chances for throwing 4 white and 2 black; the fourth $20a^3b^3$ will express the number of Chances for throwing 3 white and 3 black; the fifth $15a^2b^4$ will express the number of Chances for throwing two white and 4 black; the sixth $6ab^3$ will express the number of Chances for 2 white and 4 black;[3] lastly, the seventh b^6 will express the number of Chances for 6 black.

And therefore having raised the Binomial $a+b$ to any given power, we may by bare inspection determine the property of any one term belonging to that power, by only observing the Indices wherewith the quantities a and b are affected in that term, since the respective numbers of white and black faces are represented by those Indices.

The better to compare the consequences that may be derived from the consideration of the Binomial $a+b$ raised to a power given, with the method of Solution that hath been

[3] De Moivre is obviously in error here.

explained before; let us resume some of the preceding questions, and see how the Binomial can be applied to them.

Suppose therefore that the Probability of throwing an Ace in four throws with a common Die of six faces be demanded.

In order to answer this, it must be considered that the throwing of one Die four times successively, is the same thing as throwing four Dice as once; for whether the same Die is used four times successively, or whether a different Die is used in each throw, the Chance remains the same; and whether there is a long or a short interval between the throwing of each of these four different Dice, the Chance remains still the same; and therefore if four Dice are thrown at once, the Chance of throwing an Ace will be the same as that of throwing it with one and the same Die in four successive throws.

This being premised, we may transfer the notion that was introduced concerning white and black faces, in the Dice, to the throwing or missing of any point or points upon those Dice, and therefore in the present case of throwing an Ace with four Dice, we may suppose that the Ace in each Die answer to one white face, and the rest of the points to five black faces, that is, we may suppose that $a = 1$, and $b = 5$; and therefore, having raised $a + b$ to its fourth power, which is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, every one of the terms wherein a is perceived will be part of te number of Chances whereby an Ace may be thrown. Now there are four of those parts into which a enters, *viz.* $a^4 + 4a^3b + 6a^2b^2 + 4ab^3$, and therefore having made $a = 1$, and $b = 5$, we shall have $1 + 20 + 150 + 500 = 671$ to express the number of Chances whereby an Ace may be thrown with four Dice, or an Ace thrown in four successive throws of one single Die: but the number of all the Chances is the fourth power of $a + b$, that is the fourth power of 6, which is 1296; and therefore the Probability required is measured by the fraction $\frac{671}{1296}$, which is conformable to the resolution given in the 3rd case of the questions belonging to the 10th Art.

It is to be observed, that the Solution would have been shorter, if instead of inquiring at first into the Probability of throwing an Ace in four throws, the Probability of its contrary, that is the Probability of missing the Ace four times successively, had been inquired into; for since this case is exactly the same as that of missing all the Aces with four Dice, and that the last term b^4 of the Binomial $a+b$ raised to its fourth power expresses the number of Chances whereby the Ace may fail in every one of the Dice; it follows, that the Probability of that failing is $\frac{b^4}{(a+b)^4} = \frac{625}{1296}$, and therefore the Probability of not failing, that is of throwing an Ace in four throws, is $1 - \frac{625}{1396} = \frac{1296-625}{1296} = \frac{671}{1296}$.

From hence it follows, that let the number of Dice be what it will, suppose n, then the last term of the power $(a + b)^n$, that is b^n, will always represent the number of Chances whereby the Ace may fail n times, whether the throws be considered as successive or cotemporary: Wherefore $\frac{b^n}{(a+b)^n}$ is the Probability of that failing; and consequently the Probability of throwing an Ace in a number of throws expressed by n, will be $1 - \frac{b^n}{(a+b)^n} = \frac{(a+b)^n - b^n}{(a+b)^n}$.

Again, suppose it be required to assign the Probability of throwing with one single Die two Aces in four throws, or of throwing at once two Aces with four Dice: the question will be answered by help of the Binomial $a + b$ raised to its fourth power, which being $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the three terms $a^4 + 4a^3b + 6a^2b^2$ wherein the Indices of a equal or exceed the number of times that the Ace is to be thrown; will denote the number of Chances whereby two Aces may be thrown; wherefore having interpreted a by 1, and b by 5, the three terms above-written will become $1+20+150 = 171$, but the whole number of Chances, *viz.* $(a+b)^4$ is in this case $= 1296$, and therefore the Probability of throwing two Aces in four throws will be measured by the fraction $\frac{171}{1296}$.

But if we chuse to take at first the Probability of the contrary, it is plain that out of the five terms that the fourth power of $a+b$ consists of, the two terms $4ab^3 + b^4$; in the first of which a enters but once, and in the second of which it enters not, will express the number of Chances that are contrary to the throwing of two Aces; which number of Chances will be found equal to $500 + 625 = 1125$. And therefore the Probability of not throwing two Aces in four throws will be $\frac{1125}{1296}$: from whence may be deduced the Probability of doing it, which therefore will be $1 - \frac{1125}{1296} = \frac{1296-1125}{1296} = \frac{171}{1296}$ as it was found in the preceding paragraph; and this agrees with the Solution of the sixth Case to be seen in the 10th Article.

Universally, the last term of any power $(a+b)^n$ being b^n, and the last but one being nab^{n-1}, in neither of which a^2 enters, it follows that the two last terms of that power express the number of Chances that are contrary to the throwing of two Aces, in any number of throws denominated by n; and that the Probability of throwing two Aces will be $1 - \frac{na^{n-1}+b^n}{(a+b)^n} = \frac{(a+b)^n - nab^{n-1} - b^n}{(a+b)^n}$.

And likewise, in the three last terms of the power $(a+b)^n$, every one of the Indices of a will be less than 3, and consequently those three last terms will shew the number of Chances that are contrary to the throwing of an Ace three times in any number of Trials denominated by n: and the same Rule will hold perpetually.

And these conclusions are in the same manner applicable to the happening or failing of any other sort of Event in any number of times, the Chances for happening and failing in any particular Trial being respectively represented by a and b.

16. Wherefore we may lay down this general Maxim; that supposing two Adversaries *A* and *B* contending about the happening of an Event, whereof *A* lays a wager that the Event will happen l times in n Trials, and *B* lays to the contrary, and that the number of Chances whereby the Event may happen in any one Trial are a, and the number of Chances whereby it may fail are b, then so many of the last terms of the power $(a+b)^n$ expanded, as are represented by l, will shew the number of Chances whereby *B* may win his wager.

Again, *B* laying a wager that *A* will not win l times, does the same thing in effect as if he laid that *A* will not win above $l-1$ times; but the number of winnings and losings between *A* and *B* is n by hypothesis, they having been supposed to play n times, and therefore subtracting $l-1$ from n, the remainder $n-l+1$ will shew that *B* himself undertakes to win $n-l+1$ times; let this remainder be called p, then it will be evident that in the same manner as the last terms of the power $(a+b)^n$ expanded, *viz.* $b^n + nab^{n-1} + \frac{n}{1} \times \frac{n-1}{2} a^2 b^{n-2}$, &c. the number whereof is l, do express the number of Chances whereby *B* may be a winner, so the first terms $a^n + na^{n-1}b + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2$, &c. the number whereof is p, do express the number of Chances whereby *A* may be a winner.

17. If A and B being at play, respectively want a certain number of Games l and p of being up, and that the respective Chances they have for winning any one particular Game be in the proportion of a to b; then raising the Binomial $a+b$ to a power whose Index shall be $l+p-1$, the number of Chances whereby they may respectively win the Set, will be in the same proportion as the Sum of so many of the first terms as are expressed by p, to the Sum of so many of the last terms as are expressed by l.

This will easily be perceived to follow from what was said in the preceding Article: for when A and B respectively undertook to win l Games and p Games, we have proved that if n was the number of Games to be played between them, then p was necessarily equal to $n-l+1$, and therefore $l+p = n+1$, and $n = l+p-1$; and consequently the power to which $a+b$ is to be raised will be $l+p-1$.

Thus supposing that A wants 3 Games of being up, and B 7, that their proportions of Chances for winning any one Game are respectively as 3 to 5, and that it were required to assign the proportion of Chances whereby they may win the Set; then making $l = 3$, $p = 7$, $a = 3$, $b = 5$, and raising $a + b$ to the power denoted by $l + p - 1$, that is in this case to the 9th power, the Sum of the first seven terms will be to the Sum of the three last, in the proportion of the respective Chances whereby they may win the Set.

Now it will be sufficient in this case to take the Sum of the three last terms; for since that Sum expresses the number of Chances whereby B may win the Set, then it being divided by the 9th power of $a + b$, the quotient will exhibit the Probability of his winning; and this Probability being subtracted from Unity, the remainder will express the Probability of A's winning: but the three last terms of the Binomial $a + b$ raised to its 9th power are $b^9 + 9ab^8 + 36aab^7$, which being converted into numbers make the Sum 37812500, and the 9th power of $a + b$ is 134217728, and therefore the Probability of B's winning will be expressed by the fraction $\frac{37812500}{134217728} = \frac{9453125}{33554432}$; let this be subtracted from Unity, then the remainder $\frac{24101307}{33554432}$ will express the Probability of A's winning; and therefore the Odds of A's being up before B, are in the proportion of 24101307 to 9453125, or very near as 23 to 9: which agrees with the Solution of the 12th Case included in the 10th Article.

In order to compleat the comparison between the two Methods of Solution which have been hitherto explained, it will not be improper to propose one case more.

Suppose therefore it be required to assign the Probability of throwing one Ace and no more, with four Dice thrown at once.

It is visible that if from the number of Chances whereby one Ace or more may be thrown, be subtracted the number of Chances whereby two Aces or more may be thrown, there will remain the number of Chances for throwing one Ace and not more; and therefore having raised the Binomial $a+b$ to its fourth power, which is $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, it will plainly be seen that the four first terms express the numbers of Chances for throwing one Ace or more, and that the three first terms express the number of Chances for throwing two Aces or more; from whence it follows that the single term $4ab^3$ does alone express the number of Chances for throwing one Ace and no more, and therefore the Probability required will be $\frac{4ab^3}{(a+b)^4} = \frac{500}{1296} = \frac{125}{324}$: which agrees with the Solution of the 7th Case given in the 10th Article.

This Conclusion might also have been obtained another way: for applying what has been said in general concerning the property of any one term of the Binomial $a + b$ raised to a power given, it will thereby appear that the term $4ab^3$ wherein the indices of a and b are respectively 1 and 3, will denote the number of Chances whereby of two contending parties *A* and *B*, the first may win once, and the other three times. Now *A* who undertakes that he shall win once and no more, does properly undertake that his own Chance shall come up once, and his adversary's three times; and therefore the term $4ab^3$ expresses the number of Chances for throwing one Ace and no more.

In the like manner, if it be required to assign the Chances for throwing a certain number of Aces, and it be farther required that there shall not be above that number, then one single term of the power $(a + b)^n$ will always answer the question.

But to find that term as expeditiously as possible, suppose n to be the number of Dice, and l the precise number of Aces to be thrown; then if l be less than $\frac{1}{2}n$, write as many terms of the Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, $\frac{n-4}{5}$, &c. as there are Units in l; or if l be greater than $\frac{1}{2}n$, write as many of them as there are Units in $\frac{1}{2}n - l$; then let all those terms be multiplied together, and the product be again multiplied by $a^l b^{n-1}$; and this last product will exhibit the term expressing the number of Chances required.

Thus if it be required to assign the number of Chances for throwing precisely three Aces, with ten Dice; here l will be $= 3$, and $n = 10$. Now because l is less than $\frac{1}{2}n$, let so many terms be taken of the Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. as there are Units in 3, which terms of this particular case will be $\frac{10}{1}$, $\frac{9}{2}$, $\frac{8}{3}$; let those terms be multiplied together, the product will be 120; let this product be again multiplied by $a^l b^{n-l}$, that is (a being $= 1$, $b = 9$, $l = 3$, $n = 10$) by 6042969, and the new product will be 725156280, which consequently exhibits the number of Chances required. Now this being divided by the 10^{th} power of $a + b$, that is, in this case, by 10000000000, the quotient 0.0725156280 will express the Probability of throwing precisely three Aces with ten Dice; and this being subtracted from Unity, the remainder 0.9274843720 will express the Probability of the contrary; and therefore the Odds against throwing three Aces precisely with ten Dice are 9274843720, to 725156280, or nearly as 64 to 5.

Although we have shewn above how to determine universally the Odds of winning, when two Adversaries being at play, respectively want certain number of Games of being up, and that they have any given proportion of Chances for winning any single Game; yet I have thought it not improper here to annex a small Table, shewing those Odds, when the number of Games wanting, does not exceed six, and that the Skill of the Contenders is equal.

Games wanting.	Odds of winning.	Games wanting.	Odds of winning.	Games wanting.	Odds of winning.
$1, 2$	$3, 1$	$2, 3$	$11, 5$	$3, 5$	$99, 29$
$1, 3$	$7, 1$	$2, 4$	$26, 6$	$3, 6$	$219, 37$
$1, 4$	$15, 1$	$2, 5$	$57, 7$	$4, 5$	$163, 93$
$1, 5$	$31, 1$	$2, 6$	$120, 8$	$4, 6$	$382, 130$
$1, 6$	$63, 1$	$3, 4$	$42, 22$	$5, 6$	$638, 386$

PROBLEM VI. (pp. 47–51)

Three Gamesters A, B, C play together on this condition, that he shall win the Set who has soonest got a certain number of Games; the proportion of the Chances which each of them has to get any one Game assigned, or which is the same thing, the proportion of their skill, being respectively as a, b, c. Now after they have played some time, they find themselves in this circumstance, that A wants 1 Game of being up, B 2 Games, and C 3 Games; the whole Stake amongst them being supposed 1; what is the Expectation of each?

SOLUTION. I.

In the circumstance the Gamesters are in, the Set will be ended in 4 Games at most; let therefore $a+b+c$ be raised to the fourth power, which will be $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 + 4a^3c + 12aabc + 4b^3c + 6aacc + 12abcc + 6bbcc + 4ac^3 + 12acbb + 4bc^3 + c^4$.

The terms $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc$, wherein the dimensions of a are equal to or greater than the number of Games which A wants, wherein also the Dimensions of b and c are less than the number of Games which B and C respectively want, are intirely favourable to A, and are part of the Numerator of his Expectation.

In the same manner, the terms $b^4 + 4b^3c + 6bbcc$ are intirely favourable to B.

And likewise the terms $4b^3c + c^4$ are intirely favourable to C.

The rest of the terms are common, as favouring partly one of the Gamesters, partly one or both of the other; wherefore these Terms are so to be divided into their parts, that the parts, respectively favouring each Gamester, may be added to his Expectation.

Take therefore all the terms which are common, *viz.* $6aabb,\ 4ab^3,\ 12abbc,\ 4ac^3$, and divide them actually into their parts; that is, 1 ° , $6aabb$ into $aabb,\ abab,\ abba,\ baab,\ baba,\ bbaa$. Out of these six parts, one part only, *viz.* $bbaa$ will be found to favour B, for 'tis only in this term that two Dimensions of b are placed before one single Dimension of a, and therefore the other five parts belong to A; let therefore $5aabb$ be added to the Expectation of A, and $1aabb$ to the Expectation of B. 2 ° . Divide $4ab^3$, into its parts $abbb,\ babb,\ bbab,\ bbba$; of these parts there are two belonging to A, and the other two to B; let therefore $2ab^3$ be added to the expectation of each. 3 ° . Divide $12abbc$ into its parts; and eight of them will belong to A, and 4 to B; let therefore $8abbc$ be added to the Expectation of A, and $4abbc$ to the Expectation of B. 4 ° . Divide $4ac^3$ into its parts, three of which will be found to be favourable to A, and one to C; add therefore $3ac^3$ to the Expectation of A, and $10ac^3$ to the Expectation of C. Hence the Numerators of the several Expectations of A, B, C, will be respectively,

(1) $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc + 5aabb + 2ab^3 + 8abbc + 3ac^3$.
(2) $b^4 + 4b^3c + 6bbcc + 1aabb + 2ab^3 + 4abcc$.
(3) $4bc^3 + 1c^4 + 1ac^3$.

The common Denominator of all their Expectations being $(a + b + c)^4$.

Wherefore if $a,\ b,\ c$, are in a proportion of equality, the Odds of winning will be respectively as $57,\ 18,\ 6$, or as $19,\ 6,\ 2$.

If n be the number of all the Games that are wanting, p the number of Gamesters, and a, b, c, d, &c. the proportion of the Chances which each Gamester has respectively to win any one Game assigned; let $a + b + c + d$, &c. be raised to the power $n + 1 - p$, and then proceed as before.

REMARK.

This is one general Method of Solution. But the simpler and more common Cases may be managed with very little trouble. As,

1 ° . Let A and B want one game each, and C two games. Then the following game will either put him in the same situation as A and B, entitling him to $\frac{1}{3}$ of the Stake; of which there is 1 Chance: or will give the whole Stake to A or B; and of this there are two Chances, C's Expectation therefore is worth $\frac{1\times\frac{1}{3}+2\times 0}{3}$ (*Introd. Art.* 5)= $\frac{1}{9}$. Take this from the Stake 1, and the Remainder $\frac{8}{9}$, to be divided equally between A and B, makes the expectations of A, B, C, to be $4,\ 4,\ 1$, respectively; to the common Denominator 9.

2 ° . Let A want 1 Game, B and C two games each. Then the next Game will either give A the whole Stake; or, one of his Adversaries winning, will reduce him to the Expectation $\frac{4}{9}$, of the former Case. His present Expectation therefore is $\frac{1\times 1+2\times\frac{4}{9}}{3} = \frac{17}{27}$: and the Complement of this to Unity, *viz.* $\frac{10}{27}$, divided equally between B and C, gives the three Expectations, $17,\ 5,\ 5$, the common Denominator being 27.

3 ° . A and B wanting each a Game, let C want 3. In this Case, C has 2 Chances for 0, and 1 Chance for the Expectation $\frac{1}{9}$, of *Case* 1. That is, his Expectation is $\frac{1}{27}$; and those of A and B are $\frac{13}{27}$, each.

4 ° . Let the Games wanting to A, B, and C, be 1, 2, 3, respectively: then A winning gets the Stake 1; B winning, A is in *Case* 3, with the Expectation $\frac{13}{27}$, or C winning, he has, as in *Case* 2, the Expectation $\frac{17}{27}$. Whence his present Expectation is $\frac{1}{3} \times \left(1 + \frac{13}{27} + \frac{17}{27}\right) = \frac{57}{81}$.

Again, A winning, B gets 0; himself winning, he acquires (*Case* 3.) the Expectation $\frac{13}{27}$. And, C winning, he is in *Case* 2, with the Expectation $\frac{5}{27}$. His present Expectation therefore is $\frac{1}{3} \times \left(0 + \frac{13}{27} + \frac{5}{27}\right) = \frac{18}{81}$. Add this to the Expectation of A, which was $\frac{57}{81}$; the Sum is $\frac{75}{81}$: and the Complement of this to Unity, which is $\frac{6}{81}$, is the Expectation of C.

Or to find C's Expectation directly: A winning, C has 0; B winning, he has Expectation $\frac{1}{27}$, (*Case* 3.) and, himself winning, he has $\frac{5}{27}$, as in *Case* 2: In all, $\frac{1}{3} \times \left(0 + \frac{1}{27} + \frac{5}{27}\right) = \frac{6}{81}$.

And thus, ascending gradually through all the inferior Cases, or by the general Rule, we may compose a Table of *Odds* for 3 Gamesters, supposed of equal Skill; like that for 2 Gamesters in Art. 17th of the Introduction.

Games wanting.			Odds.			Games wanting.			Odds.			Games wanting.			Odds.		
A.	B.	C.	a.	b.	c.	A.	B.	C.	a.	b.	c.	A.	B.	C.	a.	b.	c.
1	1	2	4	4	1	1	2	3	19	6	2	2	2	4	338	338	53
1	1	3	13	13	1	1	2	4	178	58	7	2	2	5	353	353	23
1	1	4	40	40	1	1	2	5	542	179	8	2	3	3	133	55	55
1	1	5	121	121	1	1	3	4	616	82	31	2	3	4	451	195	83
1	2	2	17	5	5	1	3	5	629	87	13	2	3	5	1433	635	119
1	3	3	65	8	8	2	2	3	34	34	13		&c.			&c.	

SOLUTION II. and more *General*.

It having been objected to the foregoing Solution, that when there are several Gamesters, and the number of games wanting amongst them is considerable; the Operation must be tedious; and that there may be some danger of mistake, in separating and collecting the several parts of their Expectations, from the Terms of the Multinomial: I invented this other Solution, which was published in the VII[th] Book of my *Miscellanea Analytica*, A. D. 1730.

The Skill of the Gamesters A, B, C, &c. is now supposed to be as a, b, c, &c. respectively: and the Games they want of the Set are p, q, r, &c. Then in order to find the Chance of a particular Gamester, as of A, or his Right in the Stake 1, we may proceed as follows.

1°. Write down Unity.

2°. Write down in order all the Letters b, c, d, &c. which denote the Skill of the Gamesters, excepting only the Letter which belongs to the Gamester whose Chance you are computing; as in our Example, the Letter a is omitted.

3°. Combine the same Letters b, c, d, &c. by two's, three's, four's, &c.

4°. Of these Combinations, leave out or cancel all such as make any Gamester besides A, the winner of the Set; that is, which give to B, q Games; to C, r Games, to D, s Games, &c.

5°. Multiply the whole by a^{p-1}.

6°. Prefix to each Product the Number of its *Permutations*, that is, of the different ways in which its Letters can be written.

7°. Let all the Products that are of the same dimension, that is, which contain the same number of Letters, be collected into different sums.

8°. Let these several Sums, from the lowest dimension upwards, be divided by the Terms of this Series, S^{p-1}, S^{p}, S^{p+1}, S^{p+2}, &c. respectively: in which *Series* $S = a + b + c + d +$&c.

9°. Lastly, multiply the Sum of the Quotients by $\frac{a}{S}$, and the Product shall be the Chance or Expectation required; namely the Right of A in the Stake 1. And in the same way, the Expectation of the other Gamesters may be computed.

EXAMPLE.

Supposing $p = 2$, $q = 3$, $r = 5$; write, as directed in the Rule,

$1,\ b+c,\ bb+bc+cc,\ bbcc+bc^3+c^4,\ bbc^3+bc^4,\ bbc^4$. Multiply each term by a^{p-1}, which in our Example is a^{2-1}, or a; prefix to each Product the number of its *Permutations*, dividing at the same time the similar Sums by S^{p-1}, S^p, S^{p+1}, &c. that is by S, S^2, S^3, &c.; And the whole multiplied into $\frac{a}{S}$ will give the Expectation of A= $\frac{a}{S}$ into $\frac{a}{S}+\frac{2ab+2ac}{S^2}+\frac{3abb+6abc+3acc}{S^3}+\frac{12abbc+12abcc+4ac^3}{S^4}+\frac{30abbcc+20abc^3+5ac^4}{S^5}+\frac{60abbc^3+30abc^4}{S^6}+\frac{105abbc^4}{S^7}$.

If we now substitute for $a,\ b,\ c$, any numbers at pleasure, we shall have the answer that belongs to those supposed degrees of Skill. As if we make $a=1,\ b=1,\ c=1$; the Expectation of A will be, $\frac{1}{3}\times\left(\frac{1}{3}+\frac{4}{9}+\frac{12}{27}+\frac{28}{81}+\frac{55}{243}+\frac{90}{729}+\frac{105}{2187}\right)=\frac{1433}{2187}$. And, by like Operations, those of B and C will be $\frac{635}{2187}$ and $\frac{119}{2187}$ respectively.