# THE DOCTRINE OF CHANCES
# PROBLEMS LVIII–LXVII

ABRAHAM DE MOIVRE
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## From the Preface.

*The Problems which follow relate chiefly to the* Duration of Play*, or to the Method of determining what number of Games may probably be played out by two Adversaries, before a certain number of Stakes agreed on between them be won or lost on either side. This Subject affording a very great Variety of Curious Questions, of which every one has a degree of Difficulty peculiar to itself, I thought it necessary to divide it into several distinct Problems, and to illustrate their Solution with proper Examples.*

*Tho' these Questions may at first sight seem to have a very great degree of difficulty, yet I have some reason to believe, that the Steps I have taken to come at their Solution, will easily be followed by those who have a competent skill in Algebra, and that the chief Method of proceeding therein will be understood by those who are barely acquainted with the Elements of that Art.*

*When I first began to attempt the general Solution of the Problem concerning the Duration of Play, there was nothing extant that could give me any light into that Subject; for altho' Mr.* de Montmort*, in the first Edition of his Book, gives the Solution of this Problem, as limited to three Stakes to be won or lost, and farther limited by the Supposition of an Equality of Skill between the Adventurers; yet he having given no Demonstration of his Solution, and the Demonstration when discovered being of very little use towards obtaining the general Solution of the Problem, I was forced to try what my own Enquiry would lead me to which having been attended with Success, the result of what I found was afterwards published in my* Specimen *before mentioned.*

*All the Problems which in my* Specimen *related to the Duration of Play, have been kept entire in the following Treatise; but the Method of Solution has received some Improvements by the new Discoveries I have made concerning the Nature of those Series which result from the Consideration of the Subject; however, the Principles of that Method having been laid down in my* Specimen*, I had nothing now to do, but to draw the Consequences that were naturally deducible from them.*

## PROBLEM LVIII.
## OF THE DURATION OF PLAY.

*Two Gamesters A and B whose proportion of skill is as* b*, each having a certain number of Pieces, play together on the condition that as often as A wins a Game, B shall give him one Piece; and that as often as B wins a Game, A shall give him one Piece; and that they cease not to play till such time as either one or the other has got all the Pieces of his Adversary: now let us suppose two Spectators R and S concerning themselves about the ending of the Play, the first of them laying that the Play will be ended in a certain number*

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*of Games which he assigns, the other laying to the contrary. To find the Probability that S has of winning his wager.*

SOLUTION.

This Problem having some difficulty, and it having given me occasion to inquire into the nature of some Series naturally resulting from its Solution, whereby I have made some improvements in the Method of summing up Series, I think it necessary to begin with the simplest Cases of this problem, in order to bring the Reader by degrees to a general Solution of it.

CASE I.

Let 2 be the number of Pieces, which each Gamester has; let also 2 be the number of Games about which the Wager is laid: now because 2 is the number of Games contended for, let $a + b$ be raised to its Square, *viz.* $aa + 2ab + bb$; then it is plain that the Term $2ab$ favours S, and that the other two are against him; and consequently that the Probability he has of winning is $\frac{2ab}{(a+b)^2}$.

COROLLARY.

If $a$ and $b$ are equal, neither R or S have any Advantage or Disadvantage; but if $a$ and $b$ are unequal, R has the Advantage.

CASE II.

Let 2 be the number of Pieces of each Gamester, as before, but let 3 be the number of Games about which the Wager is laid: then $a+b$ being raised to its Cube, *viz.* $a^3 + 3aab + 3abb + b^3$, it will be seen that the two Terms $a^3$ and $b^3$ are contrary to S, they denoting the number of Chances for winning three times together; it will also be seen that the other two Terms $3aab$ and $3abb$ are partly for him, partly against him. Let therefore those two Terms be divided into their proper parts, *viz.* $3aab$ into $aab + aba + baa$, and $3abb$ into $abb + bab + bba$, and it will plainly be perceived that out of those six parts there are four which are favorable to S, *viz.* $aba$, $baa$, $abb$, $bab$ or $2aab + 2abb$; from whence it follows that the Probability which S has of winning his Wager will be $\frac{2aab+2abb}{(a+b)^3}$, or dividing both Numerator and Denominator by $a+b$, it will be found to be $\frac{2ab}{(a+b)^2}$, which is the same as in the preceding Case. The reason of which is, that the winning of a certain number of even Pieces in an odd number of Games is impossible, unless it was done in the even number of Games immediately preceding the odd number, no more than an odd number of Pieces can be won in an even number of Games, unless it was done in the odd number immediately preceding it; but still the Problem of winning an even number of Pieces in an odd number of Games is rightly proposed; for Instance, the Probability of winning either of one side or the other, 8 Pieces in 63 Games; for, provided it be done either before or at the Expiration of 62 Games, he who undertakes that it shall be done in 63 wins his Wager.

CASE III.

Let 2 be the number of Pieces of each Gamester, and 4 the number of Games upon which the Wager is laid: let therefore $a + b$ be raised to the fourth Power, which is $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4$; which being done, it is plain that the Terms $a^4 + 4a^3b + 4ab^3 + b^4$ are wholly against S, and that the only Term $6aabb$ is partly for him, and partly against him, for which reason, let this Term be divided into its parts, *viz.* $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$, and 4 of these parts, *viz.* $abab$, $abbba$, $baab$, $baba$, or $4aabb$ will be found to favour S; from which it follows that his Probability of winning will be $\frac{4aabb}{(a+b)^4}$.

## CASE IV.

If 2 be the number of Pieces of each Gamester, and 5 the number of Games about which the Wager is laid, the Probability which S has of winning his wager will be the same as in the preceding Case, *viz.* $\frac{4aabb}{(a+b)^4}$.

*Universally*, Let 2 be the number of Pieces of the Gamester, and $2+d$ the number of Games upon which the Wager is laid; and the Probability which S has of winning will be $\frac{(2ab)^{1+\frac{1}{2}d}}{(a+b)^{2+d}}$ if $d$ be an even number; or $\frac{(2ab)^{\frac{1+d}{2}}}{(a+b)^{1+d}}$ if $d$ be odd, writing $d-1$ instead of $d$.

## CASE V.

If 3 be the number of Pieces of each Gamester, and $3+d$ the number of Games upon which the Wager is laid, then the Probability which S has of winning will be $\frac{(3ab)^{1+\frac{1}{2}d}}{(a+b)^{2+d}}$ if $d$ be an even number, or $\frac{(3ab)^{\frac{1+d}{2}}}{(a+b)^{1+d}}$ if it be odd.

## CASE VI.

If the number of Pieces of each Gamester be more than 3, the Expectation of S, or the Probability there is that the Play shall not be ended in a given number of Games, may be determined in the following manner.

*A General Rule for determining what Probability there is that the Play shall not be determined in a given number of Games.*

Let $n$ be the number of Pieces of each Gamester. Let also $n+d$ be the number of Games given; raise $a+b$ to the Power $n$, then cut off the two extream Terms, and multiply the remainder by $aa+2ab+bb$: then cut off again the two Extreams, and multiply again the remainder by $aa+2ab+bb$, still rejecting the two Extreams; and so on, making as many Multiplications as there are Units in $\frac{1}{2}d$; make the last Product the Numerator of a Fraction whose Denominator let be $(a+b)^{n+d}$, and that Fraction will express the Probability required, or the Expectation of S upon a common Stake 1, supposed to be laid between R and S; still observing that if $d$ be an odd number, you write $d-1$ in its room.

## EXAMPLE I.

Let 4 be the number of Pieces of each Gamester, and 10 the number of Games given: in this Case $n=4$, $n+d=10$; wherefore $d=6$, and $\frac{1}{2}d=3$. Let therefore $a+b$ be raised to the fourth Power, and rejecting continually the extreams, let three Multiplications be made by $aa+2ab+bb$. Thus,

$$
\begin{array}{cccccc}
a^4 & +4a^3b & +6aabb & +4ab^3 & +b^4 & \\
 & aa & +2ab & +bb & & \\
\hline
 & 4a^5b & +6a^4bb & +4a^3b^3 & & \\
 & & +8a^4bb & +12a^3b^3 & +8aab^4 & \\
 & & & +4a^3b^3 & +6aab^4 & +4ab^5 \\
\hline
\end{array}
$$

| | | | | | |
|---|---|---|---|---|---|
| $+14a^4bb$ | $+20a^3b^3$ | $+14aab^4$ | | | |
| $aa$ | $+2ab$ | $+bb$ | | | |
| $14a^6bb$ | $+20a^5b^3$ | $+14a^4b^4$ | | | |
| | $+28a^5b^3$ | $+40a^4b^4$ | $+28a^3b^5$ | | |
| | | $+14a^4b^4$ | $+20a^3b^5$ | $+14aab^6$ | |
| | $48a^5b^3$ | $+68a^4b^4$ | $+48a^3b^5$ | | |
| | $aa$ | $+2ab$ | $+bb$ | | |
| | $48a^7b^3$ | $+68a^6b^4$ | $+48a^5b^5$ | | |
| | | $+96a^5b^4$ | $+136a^5b^5$ | $+96a^3b^6$ | |
| | | | $+48a^5b^5$ | $+68a^4b^6$ | $+48a^4b^7$ |
| | | $164a^6b^4$ | $+232a^5b^5$ | $+164a^4b^6$ | |

Wherefore the Probability that the Play will not be ended in 10 Games will be $\frac{164a^6b^4+232a^5b^5+164a^4b^6}{(a+b)^{16}}$, which Expectation will be reduced to $\frac{560}{1024}$, if there be an equality of Skill between the Gamesters; now this Fraction $\frac{560}{1024}$ or $\frac{35}{64}$ being subtracted from Unity, the remainder will be $\frac{29}{64}$, which will express the Probability of the Play's ending in 10 Games, and consequently it is 35 to 29 that, if two equal Gamesters play together, there will not be four Stakes lost on either side, in 10 Games.

*N.B.* The foregoing operation may be very much contracted by omitting the Letters $a$ and $b$, and restoring them after the last Multiplication; which may be done in this manner. Make $n+\frac{1}{2}d-1=p$, and $\frac{1}{2}d+1=q$; then annex to the respective Terms resulting from the last Multiplication the literal Products $a^pb^q$, $a^{p-1}b^{q+1}$, $a^{p-2}b^{q+2}$, &c.

Thus in the foregoing Example, instead of the first Multiplicand $4a^3b+6aabb+4ab^3$, we might have taken only $4+6+4$, and instead of multiplying three times by $aa+2ab+bb$, we might have multiplied only by $1+2+1$, which would have made the last Terms to have been $164+232+164$. Now since that $n=4$ and $d=6$, $p$ will be $=6$ and $q=4$, and consequently the literal Products to be annexed respectively to the Terms $164+232+164$ will be $a^6b^4$, $a^5b^5$, $a^4b^6$, which will make the Terms resulting from the last Multiplication to be $164a^6b^4+232a^5b^5+164a^4b^6$, as they had been found before.

## EXAMPLE II.

Let 5 be the number of Pieces of each Gamester, and 10 the number of Games given: let also the proportion of Skill between A and B be as 2 to 1.

Since $n=5$, and $n+d=10$, it follows that $d=5$. Now $d$ being an odd number must be supposed $=4$, so that $\frac{1}{2}d=2$: let therefore $1+1$ be raised to the fifth Power, and always rejecting the Extreams, multiply twice by $1+2+1$. thus

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | +5 | +10 | +10 | +5 | +1 | |
| | 1 | +2 | +1 | | | |
| | 5 | +10 | +10 | +5 | | |
| | | +10 | +20 | +20 | +10 | |
| | | | +5 | +10 | +10 | +5 |
| | | 20 | +35 | +35 | +20 | |

| | | | | | |
|---|---|---|---|---|---|
| 20 | +35 | +35 | +20 | | |
| | 1 | +2 | +1 | | |
| 20 | +35 | +35 | +20 | | |
| | +40 | +70 | +70 | +40 | |
| | | +20 | +35 | +35 | +20 |
| | 75 | +125 | +125 | +75 | |

Now to supply the literal Products that are wanting, let $n+\frac{1}{2}d-1$ be made $=p$, and $\frac{1}{2}d+1=q$, and the Products that are to be annexed to the numerical quantities will be $a^pb^q$, $a^{p-1}b^{q+1}$, $a^{p-2}b^{q+2}$, $a^{p-3}b^{q+3}$, &c. wherefore $n$, in this Case, being $=5$, and $d=4$, then $p$ will be $=6$, and $q=3$, it follows that the Products to be annexed in this Case be $a^6b^3$, $a^5b^4$, $a^4b^5$, $a^3b^6$, and consequently the Expectation of S will be found to be $\frac{75a^6b^3+125\,a^5b^4+125\,a^4b^5+75\,a^3b^6}{(a+b)^9}$.

*N.B.* When $n$ is an odd number, as it is in this Case, the Expectation of S will always be divisible by $a+b$. Wherefore dividing both Numerator and Denominator by $a+b$. the foregoing Expression will be reduced to

$$\frac{75a^5b^3+125\,a^4b^4+125\,a^4b^4+75\,a^3b^5}{(a+b)^8} \text{ or } 25a^3b^3\times\frac{3aa+2ab+3bb}{(a+b)^8}$$

Let now $a$ be interpreted by 2, and $b$ by 1, and the Expectation of S will become $\frac{3800}{6561}$.

## PROBLEM LIX.

*The same things being given as in the preceding Problem, to find the Expectation of R, or otherwise the Probability that the Play will be ended in a given number of Games.*

### SOLUTION.

*First*, It is plain that if the Expectation of $S$ obtained by the preceding Problem be subtracted from Unity, there will remain the Expectation of $R$.

*Secondly*, Since the Expectation of $S$ decreases continually, as the number of Games increases, and that the Terms we rejected in the former Problem being divided by $aa+2ab+bb$ are the Decrement of his Expectation; it follows that if those rejected Terms be divided continually by $aa+2ab+bb$ or $(a+b)^2$, they will be the Increment of the Expectation of $R$. Wherefore the Expectation of $R$ may be expressed by means of those rejected Terms. Thus in the second Example of the preceding Problem, the Expectation of $R$ expressed by means of the rejected Terms will be found to be

$$\frac{a^5+b^5}{(a+b)^5}+\frac{5a^6b+5ab^6}{(a+b)^7}+\frac{20a^7bb+20aab^7}{(a+b)^9},\text{ or}$$
$$\frac{a^5+b^5}{(a+b)^5}\times\overline{1+\frac{5ab}{(a+b)^2}+\frac{20aabb}{(a+b)^4}}$$

In like manner, if 6 were the number of the Pieces of each Gamester, and the number of Games were $14$, it would be found that the Expectation of $R$ would be

$$\frac{a^6+b^6}{(a+b)^6}\times\overline{1+\frac{6aa}{(a+b)^2}+\frac{27aabb}{(a+b)^4}+\frac{110a^3b^3}{(a+b)^6}+\frac{429}{(a+b)^8}a^4b^4}.$$

And if 7 were the number of Pieces of each Gamester, and the number of Games were 15, then the Expectation of $R$ would be found to be

$$\frac{a^7+b^7}{(a+b)^7}\times\overline{1+\frac{7ab}{(a+b)^2}+\frac{35aabb}{(a+b)^4}+\frac{154a^3b^3}{(a+b)^6}+\frac{637a^4b^4}{(a+b)^8}}$$

*N. B.* The number of Terms of these Series will always be equal to $\frac{1}{2}d+1$, if $d$ be an even number, or to $\frac{d+1}{2}$, if it be odd.

*Thirdly*, All the Terms of these Series have to one another certain Relations, which being once discovered, each Term of any Series resulting from any Case of this Problem, may be easily generated from the preceding ones.

Thus in the first of the two last Series, the numerical Coefficient belonging to the Numerator of each Term may be derived from the preceding, in the following manner. Let $K$, $L$, $M$ be the three last Coefficients, and let $N$ be the Coefficient of the next Term required; then it will be found that $N$ in that Series will constantly be equal to $6M-9L+2K$. Wherefore if the Term which would follow $\frac{429a^4b^4}{(a+b)^4}$ in the Case of $16$ Games given, were desired; then make $M=429$, $L=110$, $K=27$, and the following Coefficient will be found $1638$. From whence it appears that the Term itself would be $\frac{1638a^5b^5}{(a+b)^{10}}$.

Likewise, in the second of the two foregoing Series, if the Law by which each Term is related to the preceding were demanded, it might thus be found. Let $K$, $L$, $M$ be the Coefficients of the three last Terms, and $N$ the Coefficient of the Term desired; then $N$ will in that Series constantly be equal to $7M-14L+7K$, or $\overline{M-2L+K}\times 7$. Now this Coefficient being obtained, the Term to which it belongs is formed immediately.

But if the universal Law by which each Coefficient is generated from the preceding be demanded, it will be expressed as follows.

Let $n$ be the number of Pieces of each Gamester: then each Coefficient contains

$n$ times the last

$-n\times\dfrac{n-3}{2}$ times the last but one

$+n\times\dfrac{n-4}{2}\times\dfrac{n-5}{3}$ times the last but two

$-n\times\dfrac{n-5}{2}\times\dfrac{n-6}{3}\times\dfrac{n-7}{4}$ times the last but three

$+n\times\dfrac{n-6}{2}\times\dfrac{n-7}{3}\times\dfrac{n-8}{4}\times\dfrac{n-9}{5}$ times the last but four.

&c.

Thus the number of Pieces of each Gamester being 6, the first Term $n$ would be $=6$, the second Term $n\times\frac{n-3}{2}$ would be $=9$, the third Term $n\times\frac{n-4}{2}\times\frac{n-5}{3}$ would be $=2$. The rest of the Terms vanishing in this Case. Wherefore if $K$, $L$, $M$ are the three last Coefficients, the Coefficient of the following Term will be $6M-9L+2K$.

*Fourthly*, The Coefficient of any Term of these Series may be found independently from any relation they may have to the preceding: in order to which, it is to be observed that each Term of these Series is proportional to the Probability of the Play's ending in a certain number of Games precisely: thus in the Series which expresses the Expectation of $R$, when each Gamester is supposed to have 6 Pieces; *viz.*

$$\frac{a^6+b^6}{(a+b)^6}\times\overline{1+\frac{6ab}{(a+b)^2}+\frac{27aabb}{(a+b)^4}+\frac{110a^3b}{(a+b)^6}+\frac{429a^4b^4}{(a+b)^8}}$$

the last Term being multiplied by the common Multiplicator $\frac{a^6+b^6}{(a+b)^6}$ set down before the Series, the Product $\frac{429a^4b^4\times(a^6+b^6)}{(a+b)^{14}}$ will denote the Probability of the Play's ending in $14$ Games precisely. Wherefore if that Term were desired which expresses the Probability of the Play's ending in 20 Games precisely, or in any number of Games denoted by $n+d$, I say that the Coefficient of that Term will be

$$\frac{n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}, \text{ \&c. continued}$$

to so many Terms as there are Units in $\frac{1}{2}d$.

$$-\frac{3n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}, \text{ \&c. continued}$$

to so many Terms as there are Units in $\frac{1}{2}d - n$.

$$+\frac{5n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}, \text{ \&c. continued}$$

to so many Terms as there are Units in $\frac{1}{2}d - 2n$.

$$-\frac{7n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}, \text{ \&c. continued}$$

to so many Terms as there are Units in $\frac{1}{2}d - 3n$.

&c.

Let now $n + d$ be supposed $= 20$, $n$ being already supposed $= 6$, then the Coefficient demanded will be found from the general Rule to be

$$\frac{6}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} = 23256$$

$$-\frac{18}{1} = -18$$

Wherefore the Coefficient demanded will be $23256 - 18 = 23238$, and then the Term itself to which this Coefficient does belong, will be $\frac{23238a^7b^7}{(a+b)^{14}}$, and consequently the Probability of the Play's ending in 20 Games precisely will be $\frac{a^6+b^6}{(a+b)^6} \times \frac{23238a^7b^7}{(a+b)^{14}}$.

But some things are to be observed about this formation of the Coefficients, which are,

*First*, that whenever it happens that $\frac{1}{2}d$, or $\frac{1}{2}d-n$, or $\frac{1}{2}d-2n$, or $\frac{1}{2}d-3n$, &c. expressing respectively the number of Multiplicators to be taken in each Line, are $= 0$, then 1 ought to be taken to supply that Line.

*Secondly*, That whenever it happens that those quantities $\frac{1}{2}d$, or $\frac{1}{2}d - n$, or $\frac{1}{2}d - 2n$, or $\frac{1}{2}d - 3n$, &c. are less than nothing, otherwise that they are negative, then the Line to which they belong, as well as all the following, ought to be cancelled.

## PROBLEM LX.

*Supposing A and B to play together till such time as four Stakes are won or lost on either side; what must be their proportion of Skill, otherwise what must be their proportion of Chances for winning any one Game assigned, to make it as probable that the Play will be ended in four Games as not?*

## SOLUTION.

The Probability of the Play's ending in four Games is by the preceding Problem $\frac{a^4+b^4}{(a+b)^4} \times 1$: now because, by Hypothesis, it is to be an equal Chance whether the Play ends or ends not in four Games; let this Expression of the Probability be made $= \frac{1}{2}$, then we shall have the Equation $\frac{a^4+b^4}{(a+b)^4} = \frac{1}{2}$: which, making $b$, $a$ :: $1, z$, is reduced to $\frac{z^4+1}{(z+1)^4} = \frac{1}{2}$, or

$z^4 - 4z^3 - 6zz - 4z + 1 = 0$. Let $12zz$ be added on both sides of the Equation, then will $z^4 - 4z^3 + 6zz - 4z + 1$ be $= 12zz$, and extracting the Square-root on both sides, it will be reduced to this *quadratic* Equation, $zz - 2z + 1 = z\sqrt{12}$, of which the two Roots are $z = 5.274$ and $z = \frac{1}{5.274}$. Wherefore whether the Skill of A be to that of B, as $5.274$ to 1, or as $1$ to $5.274$, there will be an Equality of Chance for the Play to be ended or not ended in four Games.

## PROBLEM LXI.

*Supposing that A and B play till such time as four Stakes are won or lost: What must be their proportion of Skill to make it a Wager of three to one, that the Play will be ended in four Games?*

### SOLUTION.

The Probability of the Play's ending in four Games arising from the number of Games $4$, from the number of Stakes $4$, and from the proportion of Skill, *viz.* of $a$ to $b$, is $\frac{a^4+b^4}{(a+b)^4}$; the same Probability arising from the Odds of three to one, is $\frac{3}{4}$: Wherefore $\frac{a^4+b^4}{(a+b)^4} = \frac{3}{4}$, and supposing $b,\ a :: 1,\ z$, that Equation will be changed into $\frac{z^4+1}{(z+1)^4} = \frac{3}{4}$ or $z^4 - 12z^3 + 38zz - 12z + 1 = 56zz$, and extracting the Square Root on both sides, $zz - 6z + 1 = z\sqrt{56}$, the Roots of which Equation will be found to be $13.407$ and $\frac{1}{13.407}$: Wherefore if the Skill of either one be to that of the other as $13.407$ to $1$, 'tis a Wager of three to one, that the Play will be ended in $4$ Games.

## PROBLEM LXII.

*Supposing that A and B play till such time as four Stakes are won or lost; What must be their proportion of Skill to make it an equal Wager that the Play will be ended in six Games?*

### SOLUTION.

The Probability of the Play's ending in six Games, arising from the given number of Games $6$, from the number of Stakes $4$, and from the proportion of Skill $a$ to $b$, is $\frac{a^4+b^4}{(a+b)^4} \times \frac{1+4ab}{(a+b)^2}$; the same Probability arising from an equality of Chance, is $= \frac{1}{2}$, from whence results the Equation $\frac{a^4+b^4}{(a+b)^4} \times \frac{1+4ab}{(a+b)^2} = \frac{1}{2}$, which making $b,\ a :: 1,\ z$ must be changed into the following $z^6 + 6z^5 - 13z^4 - 20z^3 - 13zz + 6z + 1 = 0$.

In this Equation, the Coefficients of the Terms equally distant from the Extreams, being the same, let it be supposed that the Equation is generated from the Multiplication of two other Equations of the same nature, *viz.* $zz - yz + 1 = 0$, and $z^4 + pz^3 + qzz + pz + 1 = 0$. Now the Equation resulting from the Multiplication of those two will be

$$\begin{array}{l} z^6 - yz^5 + 1z^4 + 2pz^3 + pz + 1 = 0. \\ \quad\ + pz^5 - pyz^4 - qyz^3 - yz \\ \qquad\qquad + qz^4 \end{array}$$

which being compared with the first Equation, we shall have $p - y = 6$, $1 - py + q = -13$, $2p - qy = -20$, from whence will be deduced a new Equation, *viz.* $y^3 + 6yy - 16y - 32 = 0$, of which one of the Roots will be $2.9644$, and this being substituted in the Equation $zz - yz + 1 = 0$, we shall at last come to the Equation $zz - 2.9644z + 1 = 0$, of which the two Roots will be $2.576$ and $\frac{1}{2.576}$; it follows therefore that if the Skill of either

Gamester be to that of the other as $2.576$ to $1$, there will be an equal Chance for four Stakes to be lost or not to be lost, in six Games.

### COROLLARY.

If the Coefficients of the extream Terms of an Equation, and likewise the Coefficients of the other Terms equally distant from the Extreams be the same, that Equation will be reducible to another, in which the Dimensions of the highest Term will not exceed half the Dimensions of the highest Term in the former.

## PROBLEM LXIII.

*Supposing A and B whose proportion of Skill is as $a$ to $b$, to play together till such time as A either wins a certain number $q$ of Stakes, or B some other number $p$ of them: what is the Probability that the Play will not be ended in a given number of Games $(n)$?*

### SOLUTION.

Multiply the Binomial $a+b$ so many times by it self as there are Units in $n-1$, always observing after every Multiplication to reject those Terms in which the Dimensions of the Quantity $a$ exceed the Dimensions of the Quantity $b$, by $q$; as also those Terms in which the Dimensions of the Quantity $b$ exceed the Dimensions of the Quantity $a$, by $p$; then shall the last Product be the Numerator of a Fraction expressing the Probability required, of which Fraction the Denominator must be the Binomial $a+b$ raised to that Power which is denoted by $n$.

### EXAMPLE.

Let $b$ be $= 3$, $q = 2$, and let the given number of Games be $= 7$. Let now the following operation be made according to the foregoing Directions.

$$a + b$$
$$a + b$$
$$\overline{aa + 2ab} + bb$$
$$a + b$$
$$\overline{2aab + 3abb} + b^3$$
$$a + b$$
$$\overline{2a^3b + 5aabb} + 3ab^3$$
$$a + b$$
$$\overline{5a^3bb + 8aab^3} + 3ab^4$$
$$a + b$$
$$\overline{5a^4bb + 13}a^3b^3 + 8aab^4$$
$$a + b$$
$$\overline{13a^4b^3 + 21a^3b^4} + 8ab^5$$

From this Operation we may conclude, that the Probability of the Play's not ending in 7 Games is equal to $\frac{13a^4b^3+21a^3b^4}{(a+b)^7}$. Now if an equality of Skill be supposed between A and B, the Expression of this Probability will be reduced to $\frac{13+21}{128}$ or $\frac{17}{64}$: Wherefore the Probability of the Play's ending in 7 Games will be $\frac{47}{64}$; from which it follows that it is $47$ to 17 that, in seven Games, either A wins two Stakes of B, or B wins three Stakes of A.

## PROBLEM LXIV.

*The same things being supposed as in the preceding Problem, to find the Probability of the Play's ending in a given number of Games.*

## SOLUTION.

*First*, If the Probability of the Play's not ending in the given number of Games, which we may obtain from the preceding Problem, be subtracted from Unity, there will remain the Probability of its ending in the same number of Games.

*Secondly*, This Probability may be expressed by means of the Terms rejected in the Operation belonging to the preceding Problem: Thus if the number of Stakes be 3 and 2, the Probability of the Play's ending in 7 Games may be expressed as follows.

$$\frac{aa}{(a+b)^2} \times \overline{1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4}}$$
$$\frac{b^3}{(a+b)^3} \times \overline{1 + \frac{3ab}{(a+b)^2} + \frac{8aabb}{(a+b)^4}}$$

Supposing both $a$ and $b$ equal to Unity, the Sum of the first Series will be $= \frac{29}{64}$, and the Sum of the second will be $\frac{18}{64}$; which two Sums being added together, the aggregate $\frac{47}{64}$ expresses the Probability that, in seven Games, either A shall win two Stakes of B, or B three Stakes of A.

*Thirdly*, The Probability of the Play's ending in a certain number of Games is always composed of a double Series, when the Stakes are unequal: which double Series is reduced to a single one, in the Case of an Equality of Stakes.

The first Series always expresses the Probability there is that A, in a given number of Games, or sooner, may win of B the number $q$ of Stakes, excluding the Probability there is that B before that time may have been in a circustance of winning the number $p$ of Stakes; both which Probabilities are not inconsistent together: for A, in fifteen Games for Instance or sooner, may win two Stakes of B, though B before that time may have been in a circumstance of winning three Stakes of A.

The second Series always expresses the Probability there is that B, in that given number of Games, may win of A a certain number $p$ of Stakes, excluding the Probability there is that A, before that time, may win of B the number $q$ of Stakes.

The first Terms of each Series may be represented respectively by the following Terms.

$$\frac{aq}{(a+b)^q} \times \left[1 + \frac{qab}{a+b} + \frac{q.q+3.aabb}{1.2.(a+b)^4} + \frac{q.q+4.q+5.a^3b}{1.2.3.(a+b)^5} \right.$$
$$\left. + \frac{q.q+5.q+6.q+7.a^4b^4}{1.2.3.4.(a+b)^8}, \text{ \&c.}\right]$$
$$\frac{b^p}{(a+b)^p} \times \left[1 + \frac{pab}{(a+b)^2} + \frac{p.p+3.aabb}{1.2.(a+b)^4} + \frac{p.p+4.p+5.a^3b^3}{1.2.3.(a+b)^6} \right.$$
$$\left. + \frac{p.p+5.p+7.a^4b^4}{1.2.3.4.(a+b)^4}, \text{ \&c.}\right]$$

Each of these Series continuing in that regularity till such time as there be a number $p$ of Terms taken in the first, and a number $q$ of Terms taken in the second; after which the Law of the continuation breaks off.

Now in order to find any of the Terms following in either of these Series, proceed thus: let $p+q-2$ be called $l$; let the Coefficient of the Term desired be $T$; let also the Coefficients of the preceding Terms taken in an inverted order, be $S,\ R,\ Q,\ P,$ &c. then will $T$ be equal to $lS - \frac{l-1}{2} \times \frac{l-2}{2} R + \frac{l-2}{1} \times \frac{l-3}{2} \times \frac{l-4}{3} Q - \frac{l-3}{1} \times \frac{l-4}{2} \times \frac{l-5}{3} \times \frac{l-6}{4} P$, &c. Thus if $p$ be $= 3$ and $q = 2$, then $l$ will be $3 + 2 - 2 = 3$, wherefore $lS - \frac{l-1}{1} \times \frac{l-2}{2} R$ would in this Case be equal to $3S - R$, which shews that the Coefficient of any Term desired would be three times the last, *minus* once the last but one.

To apply this, let it be required to find what Probability there is that in fifteen Games or sooner, either A shall win two Stakes of B, or B three Stakes of A; or which is all one, to find what Probability there is that the Play shall end in fifteen Games at farthest; A and B resolving to play till such time as A either wins two Stakes or B three.

Let 2 and 3, in the two foregoing Series, be substituted respectively in the room of $q$ and $p$, the three first Terms of the first Series will be, setting aside the common Multiplicator, $1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4}$: likewise the two first Terms of the second will be $1 + \frac{3ab}{(a+b)^2}$. Now because the Coefficient of any Term desired in each Series is respectively three times the last, *minus* once the last but one, it follows that the next Coefficient in the first Series will be found to be 13, and by the same Rule the next to it 34, and so on. In the same manner, the next Coefficient in the second Series will be found to be 8, and the next to it 21, and so on. Wherefore restoring the common Multiplicators the two Series will be

$$\frac{a^2}{(a+b)^2} \times \left[1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4} + \frac{13a^3b^3}{(a+b)^6} + \frac{34a^4b^4}{(a+b)^8} + \frac{80a^5b^5}{(a+b)^{10}} + \frac{233a^6b^6}{(a+b)^{12}}\right]$$

$$\frac{b^3}{(a+b)^3} \times \left[1 + \frac{3ab}{(a+b)^2} + \frac{8aabb}{(a+b)^4} + \frac{21a^3b^3}{(a+b)^6} + \frac{55a^4b^4}{(a+b)^8} + \frac{144a^5b^5}{(a+b)^{10}} + \frac{377a^6b^6}{(a+b)^{12}}\right]$$

If we suppose an equality of Skill between A and B, the Sum of the first Series will be $\frac{18778}{32768}$, the Sum of the second will be $\frac{12393}{32768}$, and the Aggregate of those two Sums will be $\frac{31171}{32768}$, which will express the Probability of the Play's ending in fifteen Games or sooner. This last Fraction being subtracted from Unity, there will remain $\frac{1597}{32768}$, which expresses the Probability of the Play's continuing beyond fifteen Games: Wherefore 'tis 31171 to 1597, or 39 to 2 nearly that one of the two equal Gamesters that shall be pitched upon, shall in fifteen Games at farthest, either win two Stakes of his Adversary, or lose three to him.

*N. B.* The Index of the Denominator in the last Term of each Series, and the Index of the common Multiplicator prefixed to it being added together, must either equal the number of Games given, or be less than it by Unity. Thus in the first Series, the Index 12 of the Denominator of the last Term, and the Index 2 of the common Denominator being added together, the Sum is 14, which is less by Unity than the number of Games given. So likewise in the second Series, the Index 12 of the Denominator of the last Term, and the Index 3 of the common Multiplicator being added together, the Sum is 15, which precisely equals the number of Games given.

It is carefully to be observed that those two Series taken together express the Expectation of one and the same person, and not of two different persons; that is properly of a Spectator, who lays a wager that the Play will be ended in a given number of Games. Yet in one Case, they may express the Expectations of two different persons: for Instance, of the Gamesters themselves, provided that both Series be continued indefinitely; for in that Case, the first Series infinitely continued will express the Probability that the Gamester A may sooner win two Stakes of B, than that he may lose three to him: likewise the second Series infinitely continued will express the Probability that the Gamester B may sooner win three Stakes of A, than lose two to him. And it will be found, (when I come to treat of the Method of summing up this sort of Series, whose Terms have a perpetual recurrency of relation to a fixed number of preceding Terms) that the first Series infinitely continued is to the second infinitely continued, in the proportion of $aa \times \overline{aa + ab + bb}$ to $b^3 \times \overline{a + b}$; that is in the Case of an Equality of Skill as 3 to 2, which is conformable to what I have said in the IXth Problem.

*Fourthly*, Any Term of these Series may be found independently from any of the preceding: for if a Wager be laid that A shall either win a certain number of Stakes denominated by $q$, or that B shall win a certain number of them denominated by $p$, and that the number of Games be expressed by $q + d$; then I say that the Coefficient of any Term in the first Series answering to that number of Games will be

$+\frac{q}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Multiplicators as there are Units in $\frac{1}{2}d$.

$-\frac{q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - p$.

$+\frac{3q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - p - q$.

$-\frac{3q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - 2p - q$.

$+\frac{5q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - 2p - 2q$.

$-\frac{5q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - 3p - 2q$.

$+\frac{7q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued so many Terms as there are Units in $\frac{1}{2}d - 3p - 3q$.

And so on.

And the same Law will hold for the other Series, calling $p + \delta$ the number of Games given, and changing $q$ into $p$, and $p$ into $q$, as also $d$ into $\delta$, still remembering that when $d$ is an odd number, $d - 1$ ought to be taken in the room of it, and the like for $\delta$.

And the same observation must be made here as was made at the end of the LIXth Problem, *viz.* that if $\frac{1}{2}d$, or $\frac{1}{2}d - p$, or $\frac{1}{2}d - p - q$, or $\frac{1}{2}d - 2p - q$, or $\frac{1}{2}d - 2p - 2q$, &c. expressing respectively the number of Multiplicators to be taken in each Line, are $= 0$, the 1 ought to be taken for that Line, and also, that if $\frac{1}{2}d$, or $\frac{1}{2}d - p$, or $\frac{1}{2}d - p - q$, &c. are less than nothing, otherwise negative, then the Line to which they belong as well as all the following ought to be cancelled.

## PROBLEM LXV.

*If A and B, whose proportion of skill is supposed as $a$ to $b$, play together: What is the Probability that one of them, suppose A, may in a number of Games not exceeding a number given, win of B a certain number of Stakes? leaving it wholly indifferent whether B, before the expiration of those Games, may or may not have been in a circumstance of winning the same, or any other number of Stakes of A.*

## SOLUTION.

Supposing $n$ to be the number of Stakes which A is to win of B, and $n+d$ the number of Games; let $a+b$ be raised to the Power whose Index is $n+d$; then if $d$ be an odd number, take so many Terms of that Power as there are Units in $\frac{d+1}{2}$; take also so many of the Terms next following as have been taken already, but prefix to them in an inverted order, the Coefficients of the preceding Terms. But if $d$ be an even number, take so many Terms of the said Power as there are Units in $\frac{1}{2}d+1$; then take as many of the Terms next following as there are Units in $\frac{1}{2}d$, and prefix to them in an inverted order the Coefficients of the preceding Terms, omitting the last of them; and those Terms taken all together will compose the Numerator of a Fraction expressing the Probability required, the Denominator of which Fraction ought to be $(a+b)^{n+d.}$

## EXAMPLE I.

Supposing the number of Stakes which A is to win, to be *Three*, and the given number of Games to be *Ten*; let $a+b$ be raised to the tenth power, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45aab^8+10ab^9+b^{10}$. Then by reason that $n=3$, and $n+d=10$, it follows that $d$ is $=7$, and $\frac{d+1}{2}=4$. Wherefore let the Four first Terms of the said Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$, and let the four Terms next following be taken likewise without regard to their Coefficients, then prefix to them in an inverted order, the Coefficients of the preceding Terms: thus the four Terms following with their new Coefficients will be $120a^6b^4+45a^5b^5+10a^4b^6+1a^3b^7$. Then the Probability which A has of winning three Stakes of B in ten Games or sooner, will be expressed by the following Fraction

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7+120a^6b^4+45a^5b^5+10a^4b^6+a^3b^7}{(a+b)^{10}}$$

which in the Case of an Equality of Skill between A and B will be reduced to $\frac{35^2}{1024}$ or $\frac{11}{32}$.

## EXAMPLE II.

Supposing the number of Stakes which A has to win to be *Four*, and the given number of Games to be *Ten*; let $a+b$ be raised to the tenth Power, and by reason that $n$ is $=4$, and $n+d=10$, it follows that $d$ is $=6$, and $\frac{1}{2}d+1=4$; wherefore let the four first Terms of the said Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$; take also three of the Terms following, but prefix to them, in an inverted order, the Coefficients of the Terms already taken, omitting the last of them; hence the three Terms following with their new Coefficients will be $45a^6b^4+10a^5b^5+1a^4b^6$. Then the probability which A has of winning four Stakes of B in ten Games, or sooner, will be expressed by the following Fraction

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7+45a^6b^4+10a^5b^5+1a^4b^6}{(a+b)^{10}}$$

which in the Case of an Equality of Skill between A and B will be reduced to $\frac{232}{1024}$ or $\frac{29}{128}$.

## *Another* SOLUTION.

Supposing as before that $n$ be the number of Stakes which A is to win, and that the number of Games be $n+d$, the Probability which A has of winning will be expressed by the following Series

$$\frac{a^n}{(a+b)^n}\times\left[1+\frac{nab}{(a+b)^2}+\frac{n.n+3.aabb}{a.2.(a+b)^4}+\frac{n.n+4.n+5.a^3b^3}{1.2.3.(a+b)^6}\right.$$
$$\left.+\frac{n.n+5.n+6.n+7.a^4b^4}{1.2.3.4.(a+b)^8}+\text{\&c.}\right]$$

which Series ought to be continued to so many Terms as there are Units in $\frac{1}{2}d+1$; always observing to substitute $d-1$ in the room of $d$ in Case $d$ be an odd number, or which is the same thing, taking so many Terms as there are Units in $\frac{d+1}{2}$.

Now supposing, as in the first Example of the preceding Solution, that Three is the number of Stakes, and Ten the given number of Games, and also that there is an equality of Skill between A and B, the foregoing Series will become $\frac{1}{8}\times\left(1+\frac{3}{4}+\frac{9}{16}+\frac{28}{64}\right)=\frac{11}{32}$, as before.

## REMARK.

In the first attempt that I had ever made towards solving the general Problem of the Duration of Play, which was in the Year 1708, I began with the Solution of this LXVth Problem, well knowing that it might be a Foundation for what I farther wanted, since which time, by a due repetition of it, I solved the main Problem: but as I found afterwards a nearer way to it, I barely published in my first Essay on those matters, what seemed to me most simple and elegant, still preserving this Problem by me in order to be published when I should think it proper. Now in the year 1713 Mr. *de Monmort* printed as Solution of it in a Book by him published upon Chance, in which was also inserted a Solution of the same by Mr. *Nicolas Bernoulli*; and as those two Solutions seemed to me, at first sight, to have some affinity with what I had found before, I considered them with very great attention; but the Solution of Mr. *Nicolas Bernoulli* being very much crouded with Symbols, and the verbal Explication of them too scanty, I own I did not understand it thoroughly, which obliged me to consider Mr. *de Monmort's* Solution with very great attention: I found indeed that he was very plain, but to my great surprize I found him very erroneous; still in my Doctrine of Chances I printed that Solution, but rectified and ascribed it to Mr. *de Monmort*, without the least intimation of any alterations made by me; but as I had no thanks for so doing, I resume my right, and now print it as my own: but to come to the Solution.

Let it be supposed to find the number of Chances there are for A to win two Stakes of B, or for B to win three Stakes of A, in fifteen Games.

The number of Chances required is expressed by two Branches of Series; all the Series of the first Branch taken together express the number of Chances there are for A to win two Stakes of B, exclusive of the number of Chances there are for B before that time, to win three Stakes of A. All the Series of the second Branch taken together express the number of Chances there are for B to win three Stakes of A, exclusive of the number of Chances there are for A before that time to win two Stakes of B.

*First* Branch of Series

| $a^{15}$ | $a^{14}b$ | $a^{13}b^2$ | $a^{12}b^3$ | $a^{11}b^4$ | $a^{10}b^5$ | $a^9b^6$ | $a^8b^8$ | $a^7b^9$ | $a^6b^{10}$ | $a^5b^{11}$ | $a^4b^{12}$ | $a^3b^{12}$ | $a^2b^{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | +15 | +105 | +455 | +1365 | +3003 | +5005 | +5005 | +3003 | +1365 | +455 | +105 | +15 | +1 |
| | | | −1 | −15 | −105 | −455 | −455 | −105 | −15 | −1 | | | |
| | | | | | +1 | +15 | +15 | +1 | | | | | |

*Second* Branch of Series

| $b^{15}$ | $b^{14}a$ | $b^{13}a^2$ | $b^{12}a^3$ | $b^{11}a^4$ | $b^{10}a^5$ | $b^9a^6$ | $b^8a^8$ | $b^7a^9$ | $b^6a^{10}$ | $b^5a^{11}$ | $b^4a^{12}$ | $b^3a^{12}$ | $b^2a^{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | +15 | +105 | +455 | +1365 | +3003 | +5005 | +5005 | +3003 | +1365 | +455 | +105 | +15 | +1 |
| | | −1 | −15 | −105 | −455 | −1365 | −455 | −105 | −15 | −1 | | | |
| | | | | | +1 | +15 | +1 | | | | | | |

The literal Quantities which are commonly annexed to the numerical ones, are here written on the top of them; which is done, to the end that each Series being contained in one Line, the dependency they have upon one another, may thereby be made more conspicuous.

The first Series of the first Branch expresses the number of Chances there are for A to win two Stakes of B, including the number of Chances there are for B before, or at the Expiration of the fifteen Games, to be in a Circumstance of winning three Stakes of A; which number of Chances may be deduced from the LXVth Problem.

The second Series of the first Branch is a part of the first, and expresses the number of Chances there are for B to win three Stakes of A, our of the number of Chances there are for A, in the first Series to win two Stakes of B. It is to be observed about this Series, *First*, that the Chances of B expressed by it are not restrained to happen in any order, that is, either before or after A has won two Stakes of B. *Secondly*, that the literal products belonging to it are the same with those of the corresponding Terms of the first Series. *Thirdly*, that it begins and ends at an Interval from the first and last Terms of the first Series equal to the number of Stakes which B is to win. *Fourthly*, that the numbers belonging to it are the numbers of the first Series repeated in order, and continued to one half of its Terms; after which those numbers return in an inverted order to the end of that Series: which is to be understood in case the number of its Terms should happen to be even; for if it should happen to be odd, then that order is to be continued to the greatest half, after which the return is made by omitting the last number. *Fifthly*, that all the Terms of it are affected with the sign *minus*.

The Third Series is part of the second, and expresses the number of Chances there are for A to win two Stakes of B, out of the number of Chances there are in the second Series for B to win three Stakes of A; with this difference, that it begins and ends at an Interval from the first and last Terms of the second Series, equal to the number of Stakes which A is to win; and that the Terms of it are all positive.

It is to be observed, that let the number of those Series be what it will, the Interval between the beginning of the first and the beginning of the second, is to be equal to the number of Stakes which B is to win; and that the Interval between the beginning of the second and the beginning of the third, is to be equal to the number of Stakes which A is to win; and that these Intervals recur alternately in the same order. It is to be observed likewise that all these Series are alternately positive and negative.

All the observations made upon the first Branch of Series belonging also to the second, it would be needless to say any thing more of them.

Now the Sum of all the Series of the first Branch, being added to the Sum of all the Series of the second, the Aggregate of these Sums will be the Numerator of a Fraction expressing the Probability of the Play's terminating in the given number of Games; of which the Denominator is the Binomial $a + b$ raised to a Power whose Index is equal to that number of Games. Thus supposing that in the Case of this Problem both $a$ and $b$ are equal to Unity, the Sum of the Series in the first Branch will be $18778$, the Sum of the Series in the second will be $12393$, and the Aggregate of both $31171$; and the Fifteenth Power of 2 being $32768$, it follows that the Probability of the Play's terminating in Fifteen Games will be $\frac{31171}{32768}$, which being subtracted from Unity, the remainder will be $\frac{1597}{32768}$: From whence we may conclude that it is a Wager of $31171$ to $1597$, that either A in Fifteen Games shall win two Stakes of B, or B win three Stakes of A: which is conformable to what was found in the LXIVth Problem.

## PROBLEM LXVI.

*To find what Probability there is that in a given number of Games A may be winner of a certain number $q$ of Stakes, and at some other time B may likewise be winner of the number $p$ of Stakes, so that both circumstances may happen.*

## SOLUTION.

Find by our LXVth Problem the Probability which A has of winning, without any limitation, the number $q$ of Stakes: find also by the LXIIIrd Problem the Probability which A has of winning that number of Stakes before B may happen tow in the number $p$; then from the first Probability subtracting the second, the remainder will express the Probability there is that both A and B may be in a circumstance of winning, but B before A. In the like manner, from the Probability which B has of winning without limitation, subtracting the Probability which he has of winning before A, the remainder will express the Probability there is that both A and B may be in a circumstance of winning, but A before B: wherefore adding these two remainders together, their Sum will express the Probability required.

Thus if it were required to find what Probability there is, that in Ten Games A may win Two Stakes of B, and that at some other time B may win Three:

The first Series will be found to be

$$\frac{aa}{(a+b)^2} \times \overline{1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4} + \frac{14a^3b^3}{(a+b)^6} + \frac{42a^4b^4}{(a+b)^8}}$$

The second Series will be

$$\frac{aa}{(a+b)^2} \times \overline{1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4} + \frac{134a^3b^3}{(a+b)^6} + \frac{34a^4b^4}{(a+b)^8}}$$

The difference of these Series being $\frac{aa}{(a+b)^2} \times \overline{\frac{a^3b^3}{(a+b)^6} + \frac{8a^4b^4}{(a+b)^8}}$ expresses the first part of the Probability required, which in the Case of an equality of Skill between the Gamesters would be reduced to $\frac{3}{256}$.

The third Series is as follows,

$$\frac{b^3}{(a+b)^3} \times \overline{1 + \frac{3ab}{(a+b)^2} + \frac{9aabb}{(a+b)^4} + \frac{28a^3b^3}{(a+b)^6}}$$

The fourth Series is

$$\frac{b^3}{(a+b)^3} \times \overline{1 + \frac{3ab}{(a+b)^2} + \frac{8aabb}{(a+b)^4} + \frac{21a^3b^3}{(a+b)^6}}$$

The difference of these two Series being $\frac{b^3}{(a+b)^3} \times \overline{\frac{aabb}{(a+b)^4} + \frac{7a^3b^3}{(a+b)^6}}$ expresses the second part of the Probability required, which in the Case of an equality of Skill would be reduced to $\frac{11}{512}$. Wherefore the Probability required would in this Case be $\frac{3}{256} + \frac{11}{512} = \frac{17}{512}$. Whence it follows, that it is a Wager of 495 to 17, or 29 to 1 very near, that in Ten Games A and B will not both be in a circumstance of winning, *viz.* A the number $q$ and B the number $p$ of Stakes. but if by the conditions of the Problem, it were left indifferent whether A or B should win the two Stakes or the three, then the Probability required would be increased, and become as follows: *viz.*

$$\frac{aa+bb}{(a+b)^2} \times \overline{\frac{a^3b^3}{(a+b)^6} + \frac{8a^4b^4}{(a+b)^8}}$$
$$+\frac{a^3+b^3}{(a+b)^3} \times \overline{\frac{aabb}{(a+b)^4} + \frac{7a^3b^3}{(a+b)^6}}$$

which, in the Case of an equality of Skill between the Gamesters, would be double to what it was before.

## PROBLEM LXVII.

*To find what Probability there is, that in a given number of Games A may win the number $q$ of Stakes; with this farther condition, that B during that whole number of Games may never have been winner of the number $p$ of Stakes.*

## SOLUTION.

From the Probability which A has of winning without any limitation the number $q$ of Stakes, subtract the Probability there is that both A and B may be winners, viz. A of the number $q$, and B of the number $p$ of Stakes, and there will remain the Probability required.

But if the conditions of the Problem were extended to this alternative, *viz.* that either A should win the number $q$ of Stakes, and B be excluded the winning of the number $p$; or that B should win the number $p$ of stakes, and A be excluded the winning of the number $q$, the Probability that either the one or the other of these two Cases may happen will easily be deduced from what we have said.

The Rules hitherto given for the Solution of Problems relating to the Duration of Play are easily practicable, if the number of Games is but small; but if that number is large, the work will be very tedious, and sometimes swell to that degree as to be in some manner impracticable: to remedy which inconveniency, I shall here give an Extract of a paper by me produced before the Royal Society, wherein was contained a Method of solving very expeditiously the chief Problems relating to this matter, by the help of a Table of Sines, of which I had before given a hint in the first Edition of my *Doctrine of Chances*, pag. 149, and 150.