Programm,

womit zu

der öffentlichen Prüfung der Zöglinge

des

Friedrichs-Werderschen Gymnasiums,

welche

Mittwoch, den 1. April 1846
Vormittags von 9, Nachmittags von $2\frac{1}{2}$ Uhr an

in

dem Hörsaale der Anstalt
(Kur-Strasse No. 52)
Statt finden wird,

die Beschützer, Gönner und Freunde
des Schulwesens und des Gymnasiums

ergebenst einladet

Karl Eduard Bonnell,
Director und Professor.

Inhalt:

Ueber die philosophische Propädeutik als Unterrichtsgegenstand auf Schulen vom Dr. Jungk.
Mémoire sur la probabilité du jeu de rencontre par Dr. Michaelis.
Schulnachrichten vom Director.

Berlin, 1846.
Gedruckt in der Nauck'schen Buchdruckerei.

Memoir on probability in the game of rencontre,*

dedicated

to the masters of the College of Werder, his old colleagues,

G. Michaelis†

1846

Although the problem known under the name of the game has already occupied some of the most distinguished mathematicians, it has not yet been considered in the generality in which we are going to treat it in this small memoir.

Problem. *We imagine in an urn* $\alpha+\beta+\gamma+\cdots+\nu=\sigma$ *balls of which* $\alpha,\beta,\gamma,\ldots\kappa,\lambda\ldots\nu$ *are marked resp. with the numbers* 1, 2, 3, ... r, $r+1\ldots n$. *We draw them successively and we ask, what is the probability, that up to drawing* r *at least* ρ *encounters will have taken place, that is that one will draw at least* ρ *times a ball marked with the rank of its drawing.*

Solution. By putting $\alpha!\beta!\ldots\nu!=\pi$, all the balls permit $\frac{\sigma!}{\pi}$ arrangements which constitute the equally possible cases. We name generally $z_{r,\rho}$ the number of cases in which the game is ended at drawing r, and $Z_{r,\rho}$ the number of cases in which it is ended up to drawing r.

I. For $\rho=1$ we have $z_{1,1}=\frac{\alpha(\sigma-1)}{\pi}$
because $z_{1,1}$ is the number of the arrangements of $\sigma-1$ elements of which resp. $\alpha-1$, β, $\gamma\ldots$ are of the same kind.

Likewise no. 2 exits at its rank in $\frac{\beta(\sigma-1)!}{\pi}$ cases from which it is necessary to subtract the number of cases in which the two first numbers exit at their rank, that is $\frac{\alpha\beta(\sigma-2)!}{\pi}$. Therefore we will have

$$z_{2,1}=\frac{\beta}{\pi}[(\sigma-1)!-\alpha(\sigma-2)!]$$
$$Z_{2,1}=z_{1,1}+z_{2,1}=\frac{1}{\pi}[(\alpha+\beta)(\sigma-1)!-\alpha\beta(\sigma-2)!].$$

No. 3 exits at its rank in $\frac{\gamma(\sigma-1)!}{\pi}$ cases from which it is necessary to subtract those in which the game is already ended up to the second drawing, that is $\frac{\gamma}{\pi}[(\alpha+\beta)(\sigma-$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 15, 2009

†*Translator's note*: Michaelis was apparently an instructor at the Friedrich-Werdersche Gymnasium of Berlin.

$2)! - \alpha\beta(\sigma-3)!]$ whence

$$z_{3,1} = \frac{\gamma}{\pi}[(\sigma-1)! - (\alpha+\beta)(\sigma-2)! + \alpha\beta(\sigma-3)!]$$
$$Z_{3,1} = \frac{1}{\pi}[(\alpha+\beta+\gamma)(\sigma-1)! - (\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-2)! + \alpha\beta\gamma(\sigma-3)!].$$

By continuing thus we find easily the formulas

$$z_{r,1} = \frac{\kappa}{\pi}[(\sigma-1)! - A_1(\sigma-2)! + A_2(\sigma-3)! \ldots \pm A_{r-1}(\sigma-r)!]$$
$$Z_{r,1} = \frac{1}{\pi}[B_1(\sigma-1)! - B_2(\sigma-2)! + B_3(\sigma-3)! \ldots \pm B_r(\sigma-r)!],$$

where $A_1,\ A_2,\ A_3,\ldots$ designating the sums of the combinations of the 1st, 2nd, 3rd... class formed from the elements $\alpha,\beta,\gamma\ldots$ which precede κ, and $B_1,\ B_2,\ B_3\ldots$ the sums of the analogous combinations formed from the elements $\alpha,\beta,\gamma\ldots\kappa$.

II. For $\rho = 2$ we have first evidently

$$z_{2,2} = \frac{\alpha\beta(\sigma-2)!}{\pi}.$$

In order that the game be ended by the third drawing it is necessary that in this drawing there exits a ball marked with no. 3 and that up to the second drawing there has been at least one encounter, but not yet two. Therefore

$$z_{3,2} = \frac{\gamma}{\pi}[(\alpha+\beta)(\sigma-2)! - \alpha\beta(\sigma-3)!] - \frac{\alpha\beta\gamma(\sigma-3)!}{\pi}$$
$$= \frac{\gamma}{\pi}[(\alpha+\beta)(\sigma-2)! - 2\alpha\beta(\sigma-3)!]$$
$$Z_{3,2} = \frac{1}{\pi}[(\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-2)! - 2\alpha\beta\gamma(\sigma-3)!]$$

In order to find likewise $z_{4,2}$ there are

$$\frac{\delta}{\pi}[(\alpha+\beta+\gamma)(\sigma-2)! - (\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-3)! + \alpha\beta\gamma(\sigma-4)!]$$

cases in which no. 4 exits at its rank, while up to the 3rd drawing there has already been at least one encounter. By subtracting those in which two encounters have already taken place, that is

$$\frac{\delta}{\pi}[(\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-3)! - 2\alpha\beta\gamma(\sigma-4)!]$$

there comes

$$z_{4,2} = \frac{\delta}{\pi}[(\alpha+\beta+\gamma)(\sigma-2)! - 2(\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-3)! + 3\alpha\beta\gamma(\sigma-4)!]$$
$$Z_{4,2} = \frac{1}{\pi}[(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\sigma-2)! - 2(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\sigma-3)! + \alpha\beta\gamma\delta(\sigma-4)!]$$

and generally

$$z_{r,2} = \frac{\kappa}{\pi}[A_1(\sigma-2)! - 2A_2(\sigma-3)! + 3A_3(\sigma-4)! \ldots \pm (r-1)A_{r-1}(\sigma-r)!]$$
$$Z_{r,2} = \frac{1}{\pi}[B_2(\sigma-2)! - 2B_3(\sigma-3)! + 3B_4(\sigma-4)! \ldots \pm (r-1)B_r(\sigma-r)!]$$

III. When $\rho = 3$, we have $z_{3,3} = \frac{\alpha\beta\gamma(\sigma-3)!}{\pi}$
In order that the 4th drawing end the game it is necessary that among the cases in which no. 4 exits at its rank, there has already been up to the 3rd drawing at least two encounters, but not yet three, whence,

$$z_{4,3} = \frac{\delta}{\pi}[(\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-3)! - 2\alpha\beta\gamma(\sigma-4)!] - \frac{1}{\pi}\alpha\beta\gamma\delta(\sigma-4)!$$
$$= \frac{\delta}{\pi}[(\alpha\beta+\alpha\gamma+\beta\gamma)(\sigma-3)! - 3\alpha\beta\gamma(\sigma-4)!]$$
$$Z_{4,3} = \frac{1}{\pi}[(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\sigma-3)! - 3\alpha\beta\gamma\delta(\sigma-4)!]$$

Among the cases relative to the hypothesis that no. 5 exits at its rank, there are

$$\frac{\epsilon}{\pi}[(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\sigma-3)! - 2(\alpha\beta\gamma+\cdots+\beta\gamma\delta)(\sigma-4)! + 3\alpha\beta\gamma\delta(\sigma-5)!]$$

in which up to the 4th drawing at least two encounters have taken place. By subtracting

$$\frac{\epsilon}{\pi}[(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\sigma-4)! - 3\alpha\beta\gamma\delta(\sigma-5)!]$$

we have

$$z_{5,3} = \frac{\epsilon}{\pi}[(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)(\sigma-3)! - 3(\alpha\beta\gamma+\cdots+\beta\gamma\delta)(\sigma-4)! + 6\alpha\beta\gamma\delta(\sigma-5)!]$$
$$Z_{5,3} = \frac{1}{\pi}[(\alpha\beta\gamma+\alpha\beta\delta+\cdots+\epsilon\gamma\delta)(\sigma-3)! - 3(\alpha\beta\gamma\delta+\alpha\beta\gamma\epsilon+\cdots+\beta\gamma\delta\epsilon)(\sigma-4)! + 6\alpha\beta\gamma\delta\epsilon(\sigma-5)!]$$

and therefore in order

$$z_{r,3} = \frac{\kappa}{\pi}[A_2(\sigma-3)! - 3_2A_3(\sigma-4)! + 4_2A_4(\sigma-5)! \ldots \pm (r-1)_2A_{r-1}(\sigma-r)!]$$
$$Z_{r,3} = \frac{1}{\pi}[B_3(\sigma-3)! - 3_2B_4(\sigma-4)! + 4_2B_5(\sigma-5)! \ldots \pm (r-1)_2B_r(\sigma-r)!],$$

where $3_2 = \frac{3.2}{2}$, $4_2 = \frac{4.3}{2}$ etc.
By applying always the same principles, there comes finally

$$z_{r,\rho} = \frac{\kappa}{\pi}[A_{\rho-1}(\sigma-\rho)! - \rho_{\rho-1}A_\rho(\sigma-\rho-1)! + (\rho+1)_{\rho-1}A_{\rho+1}(\sigma-\rho-2)! \ldots \pm (r-1)_{\rho-1}A_{r-1}(\sigma-r)!]$$
$$Z_{r,\rho} = \frac{1}{\pi}[B_\rho(\sigma-\rho)! - \rho_{\rho-1}B_{\rho+1}(\sigma-\rho-1)! + (\rho+1)_{\rho-1}B_{\rho+2}(\sigma-\rho-2)! \ldots \pm (r-1)_{\rho-1}B_r(\sigma-r)!]$$

In order to find these formulas, we suppose, that they are verified for the values $1, 2, 3, \ldots r$ and $1, 2, 3, \ldots \rho$ of r and of ρ and we demonstrate, that they subsist yet, if we increase r or ρ by unity.

We seek $z_{r+1,\rho}$. Among the cases in which $r+1$ exits at its rank, the number of those in which there are at least $\rho-1$ encounters up to drawing r, is

$$\frac{\lambda}{\pi}[B_{\rho-1}(\sigma-\rho)!-(\rho-1)_{\rho-2}B_\rho(\sigma-\rho-1)!+\rho_{\rho-2}B_{\rho+1}(\sigma-\rho-2)!\ldots\pm(r-1)_{\rho-2}B_r(\sigma-r-1)!]$$

It is necessary to subtract from this number the cases in which there are less than ρ encounters, that is

$$\frac{\lambda}{\pi}[B_\rho(\sigma-\rho-1)!-\rho_{\rho-1}B_{\rho+1}(\sigma-\rho-2)!\ldots\pm(r-1)_{\rho-1}B_r(\sigma-r-1)!],$$

whence

$$\begin{aligned}
z_{r+1,\rho} =& \frac{\lambda}{\pi}[B_{\rho-1}(\sigma-\rho)!-\rho_{\rho-1}B_\rho(\sigma-\rho-1)!+(\rho+1)_{\rho-1}B_{\rho+1}(\sigma-\rho-2)!\ldots\pm r_{\rho-1}B_r(\sigma-r-1)!]\\
Z_{r+1,\rho} =& \frac{1}{\pi}[(B_\rho+\lambda B_{\rho-1})(\sigma-\rho)!-\rho_{\rho-1}(B_{\rho+1}+\lambda B_\rho)(\sigma-\rho-1)!]\\
&+(\rho+1)_{\rho-1}(B_{\rho+2}+\lambda B_{\rho+1})(\sigma-\rho-2)!\ldots\pm r_{\rho-1}B_r(\sigma-r-1)!]\\
=& \frac{1}{\pi}[C_\rho(\sigma-\rho)!-\rho_{\rho-1}C_{\rho+1}(\sigma-\rho-1)!+(\rho+1)_{\rho-1}C_{\rho+2}(\sigma-\rho-2)!\ldots\pm r_{\rho-1}C_{r+1}(\sigma-r-1)!]
\end{aligned}$$

where C_ρ, $C_{\rho+1}\ldots$ designate the sums of the combinations of the classes ρ, $\rho+1\ldots$ formed from the elements $\alpha,\beta\ldots\kappa,\lambda$.

It follows, whatever be the value of ρ in the two formulas, the value of r can be increased by unity.

We prove now, that we can make ρ vary by unity. Let ξ, η, θ be the numbers of balls marked with the nos. ρ, $\rho+1$, $\rho+2$, we will have evidently

$$z_{\rho+1,\rho+1}=\frac{1}{\pi}\alpha\beta\gamma\ldots\eta(\sigma-\rho-1)!$$

In order that $\rho+1$ encounters taken place up to drawing $\rho+2$ and that the game be ended by this drawing, it is necessary to seek among the cases in which $\rho+2$ exits at its rank, those in which up to drawing $\rho+1$ at least ρ encounters have taken place. We obtain

$$\frac{\theta}{\pi}[(\alpha\beta\ldots\xi+\cdots+\beta\gamma\ldots\eta)(\sigma-\rho-1)!-\rho\alpha\beta\gamma\ldots\eta(\sigma-\rho-2)!]$$

By subtracting the $\frac{1}{\pi}\alpha\beta\gamma\ldots\eta\theta(\sigma-\rho-2)!$ in which up to the $(\rho+1)$th drawing $\rho+1$ encounters have taken place, we will have

$$z_{\rho+2,\rho+1}=\frac{\theta}{\pi}[(\alpha\beta\ldots\xi+\cdots+\beta\gamma\ldots\eta)(\sigma-\rho-1)!-(\rho+1)\alpha\beta\gamma\ldots\eta(\sigma-\rho-2)!]$$

$$Z_{\rho+2,\rho+1}=z_{\rho+2,\rho+1}+z_{\rho+1,\rho+1}=\frac{1}{\pi}[(\alpha\beta\ldots\eta+\cdots+\beta\gamma\ldots\eta\theta)(\sigma-\rho-1)!-(\rho+1)\alpha\beta\ldots\theta(\sigma-\rho-2)!],$$

this which proves the proposition for all the possible cases.

We name $v_{r,\rho}$ and $V_{r,\rho}$ the probabilities that the game be ended by the drawing r or up to this drawing, we obtain by dividing by the number of all the possible cases

$$v_{r,\rho} = \kappa\left(\frac{A_{\rho-1}}{\sigma(\sigma-1)\cdots(\sigma-\rho-1)} - \frac{\rho_{\rho-1}A_\rho}{\sigma\cdots(\sigma-\rho)} + \frac{(\rho+1)_{\rho-1}A_{\rho+1}}{\sigma\cdots(\sigma-\rho-1)}\cdots \pm \frac{(r-1)_{\rho-1}A_{r-1}}{\sigma\cdots(\sigma-r+1)}\right)$$
$$V_{r,\rho} = \frac{B_\rho}{\sigma\cdots(\sigma-\rho+1)} - \frac{\rho_{\rho-1}B_{\rho+1}}{\sigma\cdots(\sigma-\rho)} + \frac{(\rho+1)_{\rho-1}B_{\rho+2}}{\sigma\cdots(\sigma-\rho-1)}\cdots \pm \frac{(r-1)_{\rho-1}B_r}{\sigma\cdots(\sigma-r+1)}$$

The probability that up to drawing r there will be exactly ρ encounters, is evidently expressed by the formula

$$u_{r,\rho} = V_{r,\rho} - V_{r,\rho+1} = \frac{B_\rho}{\sigma\cdots(\sigma-\rho+1)} - \frac{(\rho+1)_\rho B_{\rho+1}}{\sigma\cdots(\sigma-\rho)} + \frac{(\rho+2)_\rho B_{\rho+2}}{\sigma\cdots(\sigma-\rho-1)}\cdots \pm \frac{r_\rho B_r}{\sigma\cdots(\sigma-r+1)}$$

Problem. *We imagine a number i of urns each containing σ balls marked exactly as in the preceding problem, and we draw one ball from each urn and thus in sequence. What is the probability that up to drawing r there exits at least one time from each urn one ball marked at the rank of the drawing?*

Solution: As the balls of each urn permit $\frac{\sigma!}{\pi}$ arrangements, the total number of equally possible cases is evidently $\left(\frac{\sigma!}{\pi}\right)^i$.

The number of cases in which the game is already finished by the first drawing, is likewise $z'_1 = \left(\frac{\alpha(\sigma-1)!}{\pi}\right)^i$.

No. 2 exits from all the urns at its rank in $\left(\frac{\beta(\sigma-1)!}{\pi}\right)^i$ cases from which it is necessary to subtract the $\left(\frac{\alpha\beta(\sigma-2)!}{\pi}\right)^i$ cases in which the game is ended at the first drawing. Therefore

$$z'_2 = \left(\frac{\beta}{\pi}\right)^i\left[(\sigma-1)!^i - \alpha^i(\sigma-2)!^i\right]$$
$$Z'_2 = \left(\frac{1}{\pi}\right)^i\left[(\alpha^i+\beta^i)(\sigma-1)!^i - \alpha^i\beta^i(\sigma-2)!^i\right]$$

No. 3 exits from all the urns at its rank in $\left(\frac{\gamma(\sigma-1)!}{\pi}\right)^i$ cases. By subtracting those in which the game is already ended at the second drawing, there comes

$$z'_3 = \left(\frac{\gamma}{\pi}\right)^i\left[(\sigma-1)!^i - (\alpha^i+\beta^i)(\sigma-2)!^i + \alpha^i\beta^i\gamma^i(\sigma-3)!^i\right]$$
$$Z'_3 = \left(\frac{1}{\pi}\right)^i\left[(\alpha^i+\beta^i+\gamma^i)(\sigma-1)!^i - (\alpha^i\beta^i+\alpha^i\gamma^i+\beta^i\gamma^i)(\sigma-2)!^i + \alpha^i\beta^i\gamma^i(\sigma-3)!^i\right]$$

et thus in order generally

$$z'_r = \left(\frac{\kappa}{\pi}\right)^i\left[(\sigma-1)!^i - A'_1(\sigma-2)!^i + A'_2(\sigma-3)!^i\cdots \pm A'_{r-1}(\sigma-r)!^i\right]$$
$$Z'_r = \left(\frac{1}{\pi}\right)^i\left[B'_1(\sigma-1)!^i - B'_2(\sigma-2)!^i + B'_3(\sigma-3)!^i\cdots \pm B'_r(\sigma-r)!^i\right]$$

where $B'_1,\ B'_2,\ B'_3\ \ldots$ designate the sums of the combinations of the $1,\ 2,\ 3\ \ldots$ classes formed by the elements $\alpha^i,\ \beta^i\ \ldots\ \kappa^i$, and $A'_1,\ A'_2,\ A'_3\ \ldots$ the analogous combinations in which κ^i does not enter.

The corresponding probabilities will be therefore

$$v'_r = \kappa^i \left(\frac{1}{\sigma^i} - \frac{A'_1}{\sigma^i(\sigma-1)^i} + \frac{A'_2}{\sigma^i(\sigma-1)^i(\sigma-2)^i} \cdots \pm \frac{A'_{r-1}}{\sigma^i \cdots (\sigma-r+1)^i} \right)$$

$$V'_r = \frac{B'_1}{\sigma^i} - \frac{B'_2}{\sigma^i(\sigma-1)^i} + \frac{B'_3}{\sigma^i(\sigma-1)^i(\sigma-2)^i} \cdots \pm \frac{B'_r}{\sigma^i \cdots (\sigma-r+1)^i}$$

We will consider now some special cases. Put $\alpha = \beta = \gamma \cdots = m$, we will have immediately

$$v_{r,\rho} = \frac{(r-1)_{\rho-1} m^\rho}{\sigma \cdots (\sigma-\rho+1)} - \frac{\rho_{\rho-1}(r-1)_\rho m^{\rho+1}}{\sigma \cdots (\sigma-\rho)} + \frac{(\rho+1)_{\rho-1}(r-1)_{\rho+1} m^{\rho+2}}{\sigma \cdots (\sigma-\rho-1)} \cdots \pm \frac{(r-1)_{\rho-1} m^r}{\sigma \cdots (\sigma-r+1)}$$

$$u_{r,\rho} = \frac{r_\rho m^\rho}{\sigma \cdots (\sigma-\rho+1)} - \frac{(\rho+1)_\rho r_{\rho+1} m^{\rho+1}}{\sigma \cdots (\sigma-\rho)} + \frac{(\rho+2)_\rho r_{\rho+2} m^{\rho+2}}{\sigma \cdots (\sigma-\rho-1)} \cdots \pm \frac{r_\rho m^r}{\sigma \cdots (\sigma-r+1)} = \frac{r}{\rho} v_{r,\rho}$$

$$V_{r,\rho} = \frac{r_\rho m^\rho}{\sigma \cdots (\sigma-\rho+1)} - \frac{\rho_{\rho-1} r_{\rho+1} m^{\rho+1}}{\sigma \cdots (\sigma-\rho)} + \frac{(\rho+1)_{\rho-1} r_{\rho+2} m^{\rho+2}}{\sigma \cdots (\sigma-\rho-1)} \cdots \pm \frac{(r-1)_{\rho-1} m^r}{\sigma \cdots (\sigma-r+1)}$$

The last formula can be put under the form

$$V_{r,\rho} = \frac{m^\rho r_\rho}{\sigma \cdots (\sigma-\rho+1)} \left\{ 1 - \frac{\rho(r-\rho)m}{(\rho+1)(\sigma-\rho)} + \frac{\rho(r-\rho)(r-\rho-1)m^r}{2!(\rho+2)(\sigma-\rho)(\sigma-\rho-1)} - \cdots \right\}$$

By putting again $r = n$, we obtain formula (C) of Laplace, Théorie analytic des probabilités, 3 edit., Paris 1820, pag. 222, assuming that our letters are replaced by those of Laplace.

Under the same assumption we will have

$$v'_r = \left(\frac{m}{\sigma}\right)^i - (r-1)\left(\frac{m^2}{\sigma(\sigma-1)}\right)^i + (r-1)_2 \left(\frac{m^3}{\sigma(\sigma-1)(\sigma-2)}\right)^i \cdots \pm \left(\frac{m^r}{\sigma \cdots (\sigma-r+1)}\right)^i$$

$$V'_r = r\left(\frac{m}{\sigma}\right)^i - r_2\left(\frac{m^2}{\sigma(\sigma-1)}\right)^i + r_3 \left(\frac{m^3}{\sigma(\sigma-1)(\sigma-2)}\right)^i \cdots \pm \left(\frac{m^r}{\sigma \cdots (\sigma-r+1)}\right)^i.$$

For $m = 1$, $r = n$ there comes

$$V'_n = \left(\frac{1}{n}\right)^{i-1} - \frac{1}{2!}\left(\frac{1}{n(n-1)}\right)^{i-1} + \frac{1}{3!}\left(\frac{1}{n(n-1)(n-2)}\right)^{i-1} \cdots \pm \frac{1}{n!}\left(\frac{1}{n!}\right)^{i-1}$$

this which agrees with the formula pag. 225 of the théorie analytique.

The preceding expressions transform themselves easily into integrals.

By applying the formula $i! = \int_0^\infty \mathrm{e}^{-x} x^i dx$, we have

$$\begin{aligned} v_{r,\rho} = \frac{m^\rho}{\sigma!} \int_0^\infty \mathrm{e}^{-x} \big(&(r-1)_{\rho-1} x^{\sigma-\rho} - \rho_{\rho-1}(r-1)_\rho m x^{\sigma-\rho-1} \\ &+ (\rho+1)_{\rho-1}(r-1)_{\rho+1} m^2 x^{\sigma-\rho-2} \cdots \pm (r-1)_{\rho-1} m^{r-\rho} x^{\sigma-r} \big)\, dx \\ =&(r-1)_{\rho-1} \frac{m^\rho}{\sigma!} \int_0^\infty \mathrm{e}^{-x} x^{\sigma-r}(x-m)^{r-\rho} dx; \end{aligned}$$

$$V_{r,\rho} = \frac{m^\rho}{\sigma!}\int_0^\infty \mathrm{e}^{-x}\left(r_\rho x^{\sigma-\rho} - \rho_{\rho-1}r_{\rho+1}mx^{\sigma-\rho-1} + (\rho+1)_{\rho-1}r_{\rho+2}m^2x^{\sigma-\rho-2}\right.$$
$$\left.\cdots \pm (r-1)_{\rho-1}m^{r-\rho}x^{\sigma-r}\right)dx$$
$$= \frac{m^\rho}{\sigma!}r(r-1)_{\rho-1}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}\left(\frac{x^{r-\rho}}{\rho} - \frac{r-\rho}{\rho+1}mx^{r-\rho-1} + \frac{(r-\rho)_2}{\rho+2}m^2x^{r-\rho-2}\cdots\right.$$
$$\left.\cdots \pm \frac{1}{r}m^{r-\rho}\right)dx,$$

whence by aide of the formula $\frac{1}{n} = \int_0^1 z^{n-1}dz$.

$$V_{r,\rho} = r(r-1)_{\rho-1}\frac{m^\rho}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}\int_0^1 \left(x^{r-\rho}z^{\rho-1} - (r-\rho)mx^{r-\rho-1}\, z^\rho\right.$$
$$\left. + (r-\rho)_2m^2x^{r-\rho-2}z^{\rho+1}\cdots \pm m^{r-\rho}z^{r-1}\right)dzdx$$
$$= r(r-1)_{\rho-1}\frac{m^\rho}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}\int_0^1 z^{\rho-1}(x-mz)^{r-\rho}\,dzdx$$

For $\rho = 1$, we have

$$v_{r,1} = \frac{m}{\sigma} - \frac{(r-1)m^2}{\sigma(\sigma-1)} + \frac{(r-1)_2m^3}{\sigma(\sigma-1)(\sigma-2)}\cdots \pm \frac{m^r}{\sigma\cdots(\sigma-r+1)}$$
$$= \frac{m}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}(x-m)^{r-1}dx$$

$$V_{r,1} = \frac{rm}{\sigma} - \frac{r_2m^2}{\sigma(\sigma-1)} + \frac{r_3m^3}{\sigma(\sigma-1)(\sigma-2)}\cdots \pm \frac{m^r}{\sigma\cdots(\sigma-r+1)}.$$
$$= \frac{rm}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}\int_0^1 (x-mz)^{r-1}\,dzdx$$
$$= \frac{1}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}(x^r-(x-m)^r)\,dx$$
$$= 1 - \frac{1}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}(x-m)^r\,dx$$

a result which contains formula (B) of the théorie analytique pag. 220.

$$V_{n,1} = 1 - \frac{(n-1)m}{2!(\sigma-1)} + \frac{(n-1)(n-2)m^2}{3!(\sigma-1)(\sigma-2)} - \cdots \pm \frac{m^{n-1}}{n(\sigma-1)\cdots(\sigma-n+1)}.$$

The probability that up to drawing r no encounter takes place, will be consequently

$$1 - \frac{rm}{\sigma} + \frac{r_2m^2}{\sigma(\sigma-1)}\cdots \mp \frac{m^r}{\sigma\cdots(\sigma-r+1)} = \frac{1}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-r}(x-m)^r\,dx.$$

In particular, if $m = 1$, we have

$$v_{r,1} = \frac{1}{n} - \frac{r-1}{n(n-1)} + \frac{(r-1)_2}{n(n-1)(n-2)}\cdots \pm \frac{1}{n\cdots(n-r+1)} = \frac{1}{n!}\int_0^\infty \mathrm{e}^{-x}x^{n-r}(x-1)^{r-1}\,dx$$
$$V_{r,1} = 1 - \frac{1}{\sigma!}\int_0^\infty \mathrm{e}^{-x}x^{n-r}(x-1)^r\,dx = \frac{r}{n} - \frac{r_2}{n(n-1)} + \frac{r_3}{n(n-1)(n-2)}\cdots \pm \frac{1}{n\cdots(n-r+1)}$$

If again $r = n$, there comes finally

$$v_{r,1} = \frac{1}{n}\left\{1 - 1 + \frac{1}{2!} - \cdots + \frac{1}{(n-1)!}\right\} = \frac{1}{n!}\int_0^\infty \mathrm{e}^{-x}(x-1)^{n-1}\,dx$$

$$= \frac{1}{n\mathrm{e}} - \frac{(-1)^n}{\mathrm{e}n!}\int_0^1 \mathrm{e}^x x^{n-1}\,dx$$

$$V_{n,1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots \pm \frac{1}{n!} = 1 - \frac{1}{n!}\int_0^\infty \mathrm{e}^{-x}(x-1)^n\,dx$$

$$= 1 - \frac{1}{\mathrm{e}} - \frac{(-1)^n}{\mathrm{e}n!}\int_0^1 \mathrm{e}^x x^n\,dx.$$

For some increasing values of the whole number n the last terms of these two formulas approach rapidly the limit 0 because

$$\int_0^1 \mathrm{e}^x x^n\,dx = \mathrm{e} - n\int_0^1 \mathrm{e}^x x^{n-1}\,dx \text{ is } < 1, \text{ each time where } n > 1.$$

Berlin, March 1846.