# LETTERS EXCHANGED

BETWEEN M. JEAN BERNOULLI AND M. DE MONTMORT
*ESSAY D'ANALYSE SUR LES JEUX DE HAZARD*, 2ND ED. 1713

**Note:** Bernoulli is commenting on the 1708 edition of the *Essay*. The page numbers referenced in the text are to that edition. Those in the footnotes are to the 2nd edition of 1713 in which these letters are contained. This latter edition is readily available.

*Letter of M. (Jean) Bernoulli to M. de Montmort*

From Basel 17 March 1710 SIR,

As I have received your beautiful Book[1] only a long time after your last Letter, I have well wished to defer the response until that which I had received and read it, in order to be in a state to say my sentiment of it to you. Although a flow on the eyes, by which I am often inconvenienced, prevents me from working much on the things which demand long calculations, especially in the winter time, I have not left to examine in the idle hours the principal ends of your Treatise, & to make myself, as much as the weakness of my eyes have permitted me, the calculation of the greater part of the Problems. I have found effectively many things very lovely & very curious for speculation, & utile for the usage that one can draw from them on occasions; but in order to make you part of the Remarks in particular that I have made here & there, in reading your Work, since you wished it, here they are.

The general route that you hold, which is to seek first the number of cases that such & such thing can arrive, is very sure & good; but it is from the prudence of the Calculator to not be plunged into a long and annoying calculation, by multiplying the cases more than it is necessary & further from necessity. For example *Pierre* wagers against *Paul* that of among 300 tokens (of which there are equally of whites, of blacks, & of reds) he will draw a white token; in order to know the ratio of their lots, I say that it is not necessary to say that there are 100 cases which make *Pierre* win, & 200 which make him lose; seeing evidently that because of indifference, or of equal facility, with which each color can be drawn, there are properly only three cases to consider; one for the white, one for the black, & one for the red; so that it is worth more to be attached to the diverse colors which can arrive with an equal facility, than to the number of the tokens, of which the equal multitude of each color varies not at all the lots of the Players, which is always as 1 to 2. It seems however that you have not observed this with much care. Here are some Examples of it. In your Book, *page 8*,[2] on the *Game of Pharaon*, in order to seek the lot of the Banker who holds for cards in his hands, among which the card of the Punter is one time, you make the denumeration of all 24 arrangements of the four cards, in order to take the favorables to the Banker, without Pharaon

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Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. .

[1] *Essay d'Analyse sur les Jeux de hazard.* Paris. 1708

[2] See page 80.

making reflection that they are not properly the diverse arrangements, but only the diverse situations of the card of the Punter among the others, which make the diversity of the cases; thus instead of your 24 arrangements, I have only these four variations to consider (I name $a$ the card of the Punter, & $b$ each of the others.)

$$1.\, bbba, \qquad 2.\, bbab, \qquad 3.\, babb, \qquad 4.\, abbb,$$

Of these four variations, the first is indifferent to the Banker, the 2nd & the 4th make him win, & the 3rd makes him lose; his lot will be

$$= \frac{1 \times A + 2 \times 2A + 1 \times 0}{4} = \frac{5}{4}A = A + \frac{1}{4}A$$ [3]

all as you have found. If the card of the Punter is found twice among the cards of the Banker, he will have these six variations, instead of 24 arrangements,

$$1.\, bbaa, \qquad 2.\, baba, \qquad 3.\, abba, \qquad 4.\, baab, \qquad 5.\, abab, \qquad 6.\, aabb,$$

The first, the third, & the fifth make the Banker win; the second & the fourth make him lose; & the sixth gives to him the half of the stake of the Punter: & hence the lot of the Banker will be

$$= (3 \times 2A + 2 \times 0 + 1 \times \frac{3}{2}A) : 6 = \frac{5}{4}A = A + \frac{1}{4}A,$$

again as you. If the card of the Punter is found three times among the cards of the Banker, one sees clearly that he must have as many variations, as when the card of the Punter is only one time there; because there is only to make a permutation of the letters $a$ to $b$ & $b$ to $a$.

$$1.\, baaa, \qquad 2.\, abaa, \qquad 3.\, aaba, \qquad 4.\, aaab,$$

Whence one draws therefore the lot of the Banker $= A + \frac{1}{4}A$. Through this manner to distribute the cases, one sees without pain, that any number of cards which the Banker holds expressed by $p$, if that of the Punter is found only one time there; the advantage of the Banker will be $\frac{1}{p}A$. It is scarcely otherwise if the card of the Punter is found more than one time among the cards of the Banker; because instead of all the arrangements that it would be necessary to examine, & of which the number is immense, for a mediocre number of cards; here it is necessary only to consider the number of variations of the two letters $a$ & $b$, which is always equal to the number of combinations, that the things, of which the number is the one of the card of the Punter, can be taken differently in the number of all the cards; & next from these combinations, of which the number is always much smaller than the one of all the arrangements, it will be easy to choose which make the Banker win, either entirely, or in part, & thus to determine his lot. For example: We give to the Banker six cards, among which we suppose that the card of the Punter is twice. Under this assumption I have only to examine, as you do, all the 720 arrangements that the six cards can incur, contenting myself to go over simply these fifteen possible variations, that the letter $a$ taken twice can make with the letter $b$ taken four times.

| | | | | |
|---|---|---|---|---|
| 1. $bbbbaa$ | 4. $babbba$ | 7. $bbabab$ | 10. $bbaabb$ | 13. $baabbb$ |
| 2. $bbbaba$ | 5. $abbbba$ | 8. $babbab$ | 11. $bababb$ | 14. $ababbb$ |
| 3. $bbabba$ | 6. $bbbaab$ | 9. $abbbab$ | 12. $abbabb$ | 15. $aabbbb$. |

[3] *Translator's note*: $A$ expresses the Money in the Game.

Among these fifteen variations, one counts seven of them, which give the total to the Banker, two which give to him his stake with the half of the stake of the Punter, & six others which make him lose; so that the lot of the Banker will be

$$= \frac{7\times 2A + 2\times \frac{3}{2}A + 6\times 0}{15} = \frac{17}{15}A = A + \frac{2}{15}A,$$

conformably to that which you have found. Thus likewise if the card of the Punter is three times among the six cards of the Banker, he will have only 20 ways to vary the situation of two letters $a$ & $b$ taken each three times, which being untangled will show that the lot of the Banker will be $= A + \frac{3}{20}A$. Following this principle, here are the general formulas that I have found for any number of cards that there are among the hands of the Banker, & any number of times that the card of the Punter is found, without supposing known the lot of the Banker in a number of cards expressed by $p-2$, as you have done in your general formula, as I have also found quite easily; here are, I say, mine.

Let

$$1.2.3.4\dots(p-q) = m, ;$$
$$(q+1)(q+2)(q+3)\dots p = n$$
$$(p-q+1)(p-q+2)(p-q+3)\dots p = l;$$

I say that the advantage of the Banker, if $q$ is an even number, will be expressed by this series

$$\frac{m}{2n}\times\left(1 + \frac{(q-1)q}{1.2} + \frac{(q-1)q(q+1)(q+2)}{1.2.3.4} + \frac{(q-1)q(q+1)\cdots(q+4)}{1.2.3.4.5.6} + \cdots \frac{(q-1)q(q+1)\cdots(p-2)}{1.2.3.4\dots(p-q)}\right)$$

Or else by this here

$$\frac{(q-1)q}{2l}\times(1.2.3\dots(q-2) + 3.4.5\dots q + 5.6.7\dots(q+2) + \cdots(p-q+1)(p-q+2)\dots(p-2))$$

But if $q$ is an odd number, one will have for the said advantage,

$$\frac{(q-1)m}{2n}\times\left(1 + \frac{q(q+1)}{1.2.3} + \frac{q(q+1)(q+2)(q+3)}{2.3.4.5} + \cdots \frac{q(q+1)(q+2)\cdots(p-2)}{2.3.4\dots(p-q)}\right)$$

or else

$$\frac{q(q+1)}{2l}\times(2.3.4\dots(q-1) + 4.5.6\dots(q+1) + 6.7.8\dots(q+3) + \cdots(p-q+1)(p-q+2)\dots(p-2));$$

where it is necessary to remark that the numbers $p$ & $q$ can be anything, provided that $q$ is no less great than 3. If you wished to take the pain, you can examine these general formulas, if they agree with yours that you gave for some particular cases, *pag. 24 & 25*.[4]

Besselte

Besides, that which I have said until here on the game of *Pharaon* must be understood also for the one of *Bassette*, *pag. 66* & following,[5] or similarly in order to calculate the favorable cases & the disadvantages to the Banker, one can spare the pain to pluck all the

[4]See page 97.
[5]See page 145.

possible arrangements of the cards, which are in the hands of the Banker, by employing only the variations of the two letters $a$ & $b$, as it has been done above; but we pass to others.

Geometric series

*Pag.* 32. *line* 15.[6] In speaking of the advantage to have the hand in the *Game of Lansquenet*, we say, Sir, that *one can express this advantage, only by a series composed of an infinite number of terms, which will always be diminishing, & that one could never have the precise value of the advantage of* Pierre. It seems that by writing this, you have not yet taken care, that this series goes always in geometric progression, which consequently, although continued to infinity, makes a sum that one can find quite easily by the common rules. The series, for example, that you gave *pag.* 35 is summable: therefore what need is there to approach to the exact value by adding a great number of terms? since one can find this value exactly in a moment & without pain, being precisely $\frac{1125}{4024}$ which is greater than $\frac{1}{4}$ or $\frac{4}{16}$, & consequently approaches it more than by $\frac{4}{17}$ than you have set for the approximate quantity; but it seems also you yourself see finally to perceive that these are some progressions, of which one can give the sum of all the terms; because on *page 51*, you gave the exact values of the advantage & of the disadvantage of the cutters. Here is a general formula, which expresses the advantage of the one who has the hand in any sort of game that it be, & who recommences to have the hand as many times as he wins, until he loses: Let $a$ be named the advantage that there is in each hand, $m$ the number of cases which make him win, & $n$ the total number of cases which make him lose; I say that his total advantage will be (by supposing $m : (m+n) = p$) equal to this series $a + pa + p^2a + p^3a$ &c. of which the sum is

$$= a : (1-p) = (m+n)a : n = a + ma : n.$$

One can find the same thing without recourse to a progression by Algebra; here is how.

Let $z$ be the stake of each Player, & thus the lot for the first game of the one who holds the hand will be

$$(m \times 2z + 2a \times 0) : (m+n) = 2mz : (m+n)$$

& hence his advantage

$$= 2mz : (m+n) - z = (m-n)z : (m+n) = a;$$

whence it follows that naming $a$ the advantage of each hand, one will have

$$z = (m+n)a : (m-n).$$

The stake of the Players being thus found, let $x$ be the total advantage, which consists in the right of *Pierre* to draw the cards as many times as he wins; there will be therefore, when he begins to play, $m$ cases which make him win the stake of his antagonist, the more the total advantage to continue the game namely

$$(m+n)a : (m-n) + x,$$

& $n$ cases which make him lose his stake, that is to say which make him have

$$(-m-n)a : (m-n);$$

one will have therefore this equality

$$x = (m((m+n)a : (m-n) + x) + n((-m-n)a : (m-n)) : (m+n)$$

which being reduced will give for total advantage

$$x = (m+n)a : n = a + ma : n,$$

[6]See page 107.

all as above. This view is more commodious than that by the progression, because one can thus find with an equal facility the advantage of *Pierre*, by supposing that *Pierre* having lost one time, the game does not finish yet; but one continues it to infinity, the hand passing alternately from one Player to the other: Let therefore the advantage of Pierre be $t$, who begins to have the hand; there will be $m$ cases, which give to him for gain the stake of his antagonist, the more the advantage to recommence, namely

$$(m+n)a:(m-n)+t,$$

& $n$ cases, which take off his stake, & which at the same time put them into the state where his antagonist was when one was going to begin the game, that is to say, which makes him have

$$(-m-n)a:(m+n)-t;$$

whence there results this equality

$$t=(m((m+n)a:(m-n)+t)+n((-m-n)a:(m-n)-t)):(m+n);$$

by the reduction of which one finds

$$t=(m+n)a:2n;$$

so that the advantage of *Pierre*, by supposing that the game must be continued to infinity, is only the half of the advantage which he had under the assumption that the game finishes as soon as he loses the hand. I amaze myself that you have not taken care to determine the advantage under this other assumption there, as the most natural & the most appropriate to the intention of the Players, who do not begin the game under the design to end it as soon as the one who has the first the hand will lose it, but rather to make pass the right of the hand from one Player to the other, a rather great number of times, so that the game can be counted to endure to infinity.

*Pag.* 59 *l.* 26.[7] The series that you gave here in order to determine the lot of *Pierre*, holding the hand in the game of Treize,[8] is very lovely & very curious, one draws it easily from the general formula on *page 58*.[9] I have also found this formula, with another, which has furnished me the same series, but without changing the signs, & which supposes the lots of the preceding numbers of the cards known, as you are going to see it. Let $S$ the lot of *Pierre* that one seeks, the number of cards that *Pierre* holds being expressed by $n$; $t$ the lot of *Pierre* the number of the cards being $n-1$; $s$ his lot, the number of the cards being $n-2$; $r$ his lot, when the number of the cards is $n-3$, & thus in sequence; one will have Treize

$$S=\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\cdots\frac{1}{1.2.3\ldots n}-\frac{t}{1}-\frac{s}{1.2}-\frac{r}{1.2.3}-\cdots\frac{0}{1.2.3\ldots n};$$

This can pass for a Theorem, your sequence being more proper in order to find first the value of $S$.

*Pag.* 63 *l.* 13.[10] You make $x=\frac{1}{2}(4A+S)+\frac{1}{2}A$; but you deceive yourself; it is necessary to make $x=\frac{1}{2}(4A+S-A)+\frac{1}{2}A$; & thus the advantage of *Pierre* is $\frac{1}{3}A$ & not $\frac{2}{3}A$: for the same reason *pag*. 64 *l.* 11 *at end* that which you say that the advantage of *Pierre* would be $2A+\frac{16}{57}A$, is not exact, because I find only $A+\frac{16}{57}A$. Bassette

*Pag*. 80.[11] It does not seem that Mr. PASCAL himself has included all the usage of his Problem of points

[7]See page 135.
[8]This series is $\frac{1}{1}-\frac{1}{1.2}+\frac{1}{1.2.3}-\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}-\frac{1}{1.2.3.4.5.6}$+&c.
[9]See page 134.
[10]See page 142.
[11]See Pages 32 & 244.

Table;[12] one of the most beautiful properties, of which one makes no mention here, being that the perpendicular bands express the coefficients of the powers of a binomial: because if the exponent of any power is named $p$, one will have

$$(a+b)^p = 1a^p + \frac{p}{1}a^{p-1}b + \frac{p(p-1)}{1.2}a^{p-2}b^2 + \frac{p(p-1)(p-2)}{1.2.3}a^{p-3}b^3 + \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}a^{p-4}b^4 + \&c.$$

this which I have found once by an entirely particular path, & I have communicated the demonstration of it to the late *Mr. the Marquis de* L' HOPITAL: one sees something of it in his posthumous Book,[13] at the place that you allege, *pag*. 92.

Huygen's 2nd

*Pages* 158 & 159.[14] You claim, Sir, to have resolved the second Problem of Mr. HUYGENS,[15] that which you have effected, in truth, in the sense that you gave to this Problem, which is that one must suppose that each Player having removed a black token, replaces it immediately into the pot, in order to leave to his successor the dozen tokens always complete, that which renders the Problem quite easy, & makes finding the lot of the three Players in the ratio of 9, 6, 4; as you have found. But it seems that Mr. HUYGENS has proposed this problem in another sense, which appears more natural, which is that all the times that one draws a black token, it is no longer replaced into the pot; if although the first shooter having lacked by drawing a black token, the second when he comes to draw, finds not more than eleven tokens; & the second having also lacked, the third finds no more than ten tokens; & the one here having similarly drawn a black, leaves only nine tokens to the first who must recommence to draw, & thus consecutively; the Problem being conceived in this sense, becomes a little more difficult, & renders the calculation of it longer. Try it, in order to see if you yourself agree with me; I have sent the solution according to the proposition in the Treatise *De Ratiociniis in ludo alea*, it is well twelve years; in consulting it, I find that I have written these three numbers, 77, 53, 35, for the ratio of the lots of the three Players.

*Page* 137.[16] I have found a formula which is expressed & is made understood more easily in this manner; for the determined cases,[17] the number of them is

$$= (1.2.3.4\ldots p) : (1.2.3\ldots b \times 1.2.3\ldots c \times 1.2.3\ldots d \times 1.2.3\ldots e \times \&c.)$$

that is to say = to a fraction of which the numerator is the number of arrangements of a multitude expressed by $p$, & the denominator the product of the numbers of arrangements of the multitudes expressed by $b$, $c$, $d$, $e$ &c. It is remarkable that this formula expresses correctly the method, that I have found once, for the determination of the coefficient of any term that one will wish of any polynomial raised to any power, this which was proposed to me once by Mr. LEIBNIZ, who approved strongly the solution which I have given to him of it, & found it useful in order to raise promptly a polynomial to a high power; because it is the polynomial $(t+x+y+z+\&c.)^p$ of which it is necessary to find the coefficient of

12 This is the *Arithmetic Triangle* so known, & on which Mr. Pascal has composed a treatise.

13 *Traité Analytique des Sections Coniques*. Paris 1707.

14 See page 219.

15 The problem is proposed in these terms. Three players A, B and C take 12 tokens of which 4 are white and 8 black; they play with this condition that the one will win who will have first, in choosing blindly, drawn a white token, and that A will choose first, B next, then C, then anew A and, thus in sequence, in rotation. One demands the ratio of their chances.

16 See pages 42 & 44.

17 The question is, being given any number $p$ of dice, which have each as many faces as one will wish, to find how many chances there are in order to bring forth the number $b$ of ace, $c$ of two, $d$ of three, $e$ of four, &c.

the term $t^b x^c y^d z^e$ &c. by supposing $b+c+d+e$ &c.$= p$; I say that the coefficient sought will be as above

$$(1.2.3.4 \ldots p) : (1.2.3 \ldots b \times 1.2.3 \ldots c \times 1.2.3 \ldots d \times 1.2.3 \ldots e \times \text{\&c.})$$

But in order to return to the question on the dice, & in order to know how many indeterminate cases there are, that with a certain number of dice one can bring forth so many of one kind, so many of another[18] &c. Let the value of this fraction be named $v$, it is necessary to multiply $v$ (supposing $R$ the number of faces of each die) by this sequence $R(R-1)(R-2)(R-3)$, continued until that which there are as many terms, as there are exponents, & to divide the product by 1.2.3, if there are three equal exponents, & to divide again by 1.2.3.4, if there are four of them equal, & thus in sequence if there are other equal exponents.

*Pages* 159 & 160.[19] The two Problems, that you set here as the third & the fourth, are in the Treatise of Mr. HUYGENS, the fourth & the third. For that which is of the one there,[20] that is to say of the fourth according to the order of Mr. HUYGENS, or of the third in your Book, it is well resolved, as much as *Pierre* wagers that among the 7 tokens which he is going to take, he will find of them exactly 3 whites, neither more nor less; because I find also that the lot of *Pierre* will be to the one of *Paul*, under this assumption, as 35 to 64, or as 280 to 512; but if one wishes that *Pierre* has won also when he draws the four whites, that which appeared to be the true sense of the words of Mr. HUYGENS, *among which will be 3 white*,[21] where it is necessary to supply *minimum*, as if *Pierre* committed himself to draw three whites at least, among the 7 tokens, that he takes among the 12: This sense being given to the Problem, one finds that the lots of *Pierre* & of *Paul* will be as 14 & 19. Huygen's 3rd & 4th

*Pag*. 162.[22] The proposition that you make here of Problem 5 of Mr. HUYGENS[23] gives to him an entirely different sense, & this is no longer the same Problem. In order to show you the greatest difference, I am going to give here the simple solutions of the one & of the other proposed in general. According to the conditions of Mr. HUYGENS, let the number of tokens be named $n$ that each Player takes at the beginning, $a$ the number of trials which make *Pierre* win a token from *Paul*, & $b$ the number of trials which make *Paul* win a token from *Pierre*; I say that their lots will be as $a^n$ & $b^n$; & thus for the particular case, which is in question, of 3 dice, where $n = 12$, $a = 27$ & $b = 15$; & so that $a : b = 27 : 15 = 9 : 5$, I say that the lot of *Pierre* will be to the one of *Paul* Huygen's 5th

$$= 9^{12} : 5^{12} = 282429536481 : 244140625.$$

You have not observed, I believe, that these two great numbers are nothing other than the 12th power of 9 & of 5. But according to the conditions of your proposition, we name

[18] That is to say, so many of simples, so many of doubles, so many of triples &c. One calls doubles, two dice which show one same point, triples, three dice which show one same point, &c.

[19] See pages 220 & 221.

[20] Problem 4. *One takes as above (Problem 3) 12 tokens of which 4 white and 8 black. A (Peter) wagers against B (Paul) that among 7 tokens that he will draw from them blindly, there will be found 3 white. One demands the ratio of the chance of A to that of B.*

[21] Actually, *inter quos 3 albi erunt.*

[22] See page 222.

[23] *Having taken each 12 tokens, A (Paul) and B (Peter) play with 3 dice on this condition that to each throw of 11 points, A must give a token to B, but that B must give 1 to A on each throw of 14 points, and that the one there will win who will be the first in possession of all the tokens. One finds in this case that the chance of A is to that of B as* 244140625 *is to* 282429536481. That which Mr. DE MONTMORT had, by mistake, understood as if *Jacques* having in hand 24 tokens, gave one of them to Paul, all the time that the dice bring forth 11, & one to Pierre, when they bring forth 14: & that the one there has won, who has first 12 tokens.

further $n$ the number of tokens which one of the two Players must win first in order to win the game, that is to say the half of the tokens that distributor *Jacques* holds at the beginning within his hands; let also $a$ be the number of trials favorable to *Pierre*, & $b$ the one of the trials favorable for *Paul*. I find that the lots of the two Players will be expressed by the two sums of the two halves of the terms of the binomial $a+b$ raised to the power $2n-1$: for example if $n=3$, the lots of *Pierre* & of *Paul* will be as the sum of the first three terms & the sum of the last three terms of $(a+b)^5$, that is to say,

$$a^5 + 5a^4b + 10a^3bb, \text{ \& } b^5 + 5b^4a + 10b^3a^2;$$

& thus in general supposing $2n-1=p$, these two series

$$a^p + \frac{p}{1}a^{p-1}b + \frac{p(p-1)}{1.2}a^{p-2}b^2 + \frac{p(p-1)(p-2)}{1.2.3}a^{p-3}b^3 + \text{\&c.}$$

&

$$b^p + \frac{p}{1}b^{p-1}a^1 + \frac{p(p-1)}{1.2}b^{p-2}a^2 + \frac{p(p-1)(p-2)}{1.2.3}b^{p-3}a^3 + \text{\&c.}$$

continued each to the number of terms expressed by $n$, will give the ratio of the lots of *Pierre* & of *Paul*. In your particular case where $n=12$, & $a:b=9:5$, it would be necessary to take twelve terms of each of these two series

$$9^{23} + \frac{23}{1}9^{22}5^1 + \frac{23.22}{1.2}9^{21}5^2 + \frac{23.22.21}{1.2.3}9^{20}5^3 + \frac{23.22.21.20}{1.2.3.4}9^{19}5^4 + \&_{c.}$$

&

$$5^{23} + \frac{23}{1}5^{22}9^1 + \frac{23.22}{1.2}5^{21}9^2 + \frac{23.22.21}{1.2.3}5^{20}9^3 + \frac{23.22.21.20}{1.2.3.4}5^{19}9^4 + \&_{c.}$$

this which would produce two numbers so great that the first would consist (according to my conjecture) at least 25 figures:[24] if you have desire to make the calculation, here is further work to exercise your patience: However you see the extreme difference, that there is between the two ways to propose the fifth Problem of Mr. HUYGENS: so that these are effectively two entirely different Problems, of which I have resolved each generally for you. You will perhaps be astonished that your 23 equalities furnish you nevertheless the same solution of Mr. HUYGENS, notwithstanding that you propose the Problem in a sense which makes it, as I just demonstrated to you, so different from the one of Mr. HUYGENS; but the reason for it is, because you follow effectively, in forming the inequalities, the conditions of Mr. HUYGENS, & not those of your proposition; because I see that you supposed that the lots of the two Players are the same when there is one same difference between the number of tokens that one has already won, & the number of tokens that the other has won; so that you supposed, for example, that their lots are the same, whether *Pierre* having won five tokens, *Paul* has three of them, or that *Pierre* having two tokens, *Paul* has none at all of them: it is this assumption, which is not just, being able to subsist with the Problem taken in the sense that you gave to it.

Problem of points

*Pag.* 177, *l.* 12 & 13.[25] *A who lacks the least points.* This restriction that there must be lacking to *Pierre* the least points is superfluous, the rule that you gave not being less good, when there is lacking to *Pierre* the most points; it would be even more expeditious, to suppose that *Pierre* is the one to whom there is lacking more points; because your series will have a smaller number of terms, which will be consequently rather added into a sum, than under the other assumption. As for the rest, I resolve this Problem more generally,

[24] Translator's note: We have $\sum_{k=0}^{11}\binom{23}{k}5^k9^{23-k} = 211{,}552{,}235{,}016{,}188{,}811{,}888{,}470{,}544$ and $\sum_{k=0}^{11}\binom{23}{k}9^k5^{23-k} = 18{,}033{,}457{,}870{,}792{,}683{,}593{,}750{,}000$.

[25] See page 244.

& by an expression simpler & more natural: I enunciate it thus. *Pierre & Paul play at many parts in an unequal game, where the number of cases favorable to Pierre is to the one of the cases favorable to Paul as* $a$ *to* $b$*; & after having played some time the number of parts which is lacking yet to Pierre is* $p$*, & the number of parts which is lacking to Paul is* $q$*; one demands the ratio of their lots.* Solution. Raise the binomial $a+b$ to the power $p+q-1=r$; the number of terms from it will be $p+q$; I say that the sum of the first terms of which the number is $q$, is to the sum of the rest of the terms of which the number will be $p$, as the lot of *Pierre* is to the one of *Paul*; now these two sums are as it follows:

$$a^r + \frac{r}{1}a^{r-1}b + \frac{r(r-1)}{1.2}a^{r-2}b^2 + \frac{r(r-1)(r-2)}{1.2.3}a^{r-3}b^3 + \text{\&c.}$$

continued to the number of terms expressed by $q$. And

$$b^r + \frac{r}{1}b^{r-1}a + \frac{r(r-1)}{1.2}b^{r-2}a^2 + \frac{r(r-1)(r-2)}{1.2.3}b^{r-3}a^3 + \text{\&c.}$$

continued to the number of terms expressed by $p$.

Supposing $p$ & $q$ equals, we fall into the case of the fifth Problem of Mr. HUYGENS, taken in the sense that you proposed it, pag. 162,[26] on which I have spoken to you amply above.

Duration of play

*Page* 178. *l.* 9.[27] *Here is a Table* &c. I am astonished that you have not observed the uniformity of this Table & the great facility with which one constructs a general Table, for any number of games that one plays always by reducing; because granted the number of the games in which Pierre & Paul agree to play, $n$: here is the Table of their lots.

TABLE

| | | | | |
|---|---|---|---|---|
| If Pierre has no point, | his lot is | $n+0$ | against | $n-0$ |
| one point | $\cdots$ | $n+1$ | $\cdots$ | $n-1$ |
| two points | $\cdots$ | $n+2$ | $\cdots$ | $n-2$ |
| three points | $\cdots$ | $n+3$ | $\cdots$ | $n-3$ |
| four points | $\cdots$ | $n+4$ | $\cdots$ | $n-4$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ points, | $\cdots$ | $n+n$ | $\cdots$ | $n-n$ |

You see that their lots go in arithmetic progression, the one ascending, the other descending: indeed your Table that you have made for the number of six only, accords with the general of mine; because the lot of

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | against | 5, | is the same thing as | $6+1$ | against | $6-1$ |
| 2 | $\cdots$ | 1 | $\cdots$ | $6+2$ | $\cdots$ | $6-2$ |
| 3 | $\cdots$ | 1 | $\cdots$ | $6+3$ | $\cdots$ | $6-3$ |
| 5 | $\cdots$ | 1 | $\cdots$ | $6+4$ | $\cdots$ | $6-4$ |
| 11 | $\cdots$ | 1 | $\cdots$ | $6+5$ | $\cdots$ | $6-5$ |

*Page* 181. *line* 13.[28] *They are absolutely impractical.* On the contrary, as it is a geometric progression, one can add in a moment as many terms as one wishes, by the common rules, as one finds in all the Books of arithmetic; thus the number of terms being $b$, I find that the sum of all the terms will be

$$= p(m^b - q^b) : (m-q)m^b;$$

[26] See page 222.
[27] See page 277.
[28] See page 128.

but for the rest, you do well to employ logarithms; I myself am served usefully in a similar occasion, it is quite twelve years, where it was a question to determine how much there remained of wine & of water mixed together in a barrel, which being at the beginning entirely full of wine, one would draw from it each day during a year a certain measure, by replenishing it immediately, after each extraction, with pure water. You will find the solution of this question, which is rather curious, in my Dissertation *De Nutritione*; that Mr. VARIGNON will be able to communicate to you. I made this question, in order to make understood, how one can determine the quantity of old material which remains in our bodies, mixed with the new, which come to us every day by nutrition, to repay the loss that our bodies make insensibly by continual transpiration.

But here is well enough of it, Sir & perhaps no more than it is necessary in order to cause boredom to you; it is for this that I do not speak to you of the faults, either in calculation, or of printing that I have encountered in some places. I pray you however to take in good part all that which I have said to here; it was for the love of the truth that I have believed must communicate to you my reflections with my ordinary frankness; so much more as you have demanded them of me. I do not omit to assess your Book as much as one can assess a Work, which gives honors to its Author, & which marks altogether a profound penetration of mind, & an indefatigable patience to make long & painful calculations. It would be wished that you wanted to take the pain to extend your Book, & to make it a more ample & richer work: the matter would not be lacking to you, especially if you wished to enter into morals & politics, as my Brother had commenced to make in his Work, which according to all appearances will never be. I understand with pleasure that Mathematics, despite the miseries of the war, flourishes more and more, & becomes even in honor in France. Here, in our country, we ourselves are unable to boast of the same happiness. Since the departure of Mr. HERMAN, I know no person, excepting my Nephew, & very few others, of whom it is necessary to hope of great progress in these sciences, which being considered, as not being *de pane lucrando*,[29] one neglects them, as of the things dry & little utile. The four Problems that you proposed at the end of your Treatise are curious; but the first appears to me to insoluble for the length of the calculation which it would demand, & that human life would not be sufficient to achieve. I do not understand the sense of the fourth: the second & the third appear to me treatable, although not without much pain & work, that I like better to leave to you in order to learn from you the solution, than to work a long time at the expense of my ordinary occupations, which leave me little spare time to apply myself to other things. I long to see the new Edition of the Book of Mr. NEWTON. It is a long time that he has promised me that he would send it to me, as soon as it would be printed; but since then I have heard no longer a thing of it; when you have received it, I pray you to send to me on occasion that which you have found: In awaiting, I give myself the honor to name myself,

SIR

Your very humble & very
obedient Servant
BERNOULLY

*P. S. my Nephew who gives you his reciprocal Compliments, comes to give me his Remarks (that I send to you here also) on your Book that I have lent to him: I have not yet had the time to examine them.*

[29]*Translator's note:* Concerning earning bread.

*Letter of M. de Montmort to Mr. Bernoulli*
At Montmort 15 November 1710 (pages 303–307)

I cannot express to you, Sir, how much I am sensible to the honor that you have done to examine my Work, & to wish well to communicate to me your scholarly & judicious Remarks. I was at Paris when I have received your Letter. I am remained around three months outside the state to make response to you by an infinity of distractions of diverse occupations. The leisure of the country of which I enjoy at present, has permitted me to examine with care the matters which are the subject of your Letter. Here is, Sir, a part of the ideas that it furnished me, I will set them according to the order that you have followed.

Pharaon

I agree, Sir, that I had not been able at all to enter into the examination of all the trials favorable indifferent or contrary to the Banker. I must have without doubt noticed that there were some cases of which the consideration was not at all necessary. But in the first Problem I have believed that the longest & most detailed way was the most proper to put the mind of the Reader into the path of the solution. You have been able to remark that I have some nicer & shorter methods than those, by looking at the formulas which I given to pages 24 & 25.[30] Those which you give, Sir, in order to express the advantage of the banker

$$\frac{m}{2n} \times 1 + \frac{q-1.q}{1.2} + \frac{q-1.q.q+1.q+2}{1.2.3.4} + \frac{q-1.q.q+1\ldots q+4}{1.2.3.4.5.6} + \cdots \frac{q-1.q.q+1\ldots p-2}{1.2.3.4\ldots p-q},$$

if $q$ is an even number. And

$$\frac{q-1.m}{2n} \times 1 + \frac{q.q-1}{2.3} + \frac{q.q+1.q+2.q+3}{2.3.4.5} + \cdots \frac{q.q+1.q+2\ldots p-2}{2.3.4\ldots p-q}$$

are very fine. I have one of them a long time since little different from yours, let it be that $q$ is even or odd. Here it is:

$$\frac{1}{p \times p-1} + \frac{p-q.p-q-1}{p.p-1.p-2.p-3} + \frac{p-q.p-q-1.p-q-2.p-q-3}{p.p-1.p-2.p-3.p-4.p-5} + \frac{p-q.p-q-1.p-q-2.p-q-3.p-q-4.p-q-5}{p.p-1.p-2.p-3.p-4.p-5.p-6.p-7} + \text{ \&c.}$$

The whole multiplied by $q.q-1.\frac{1}{2}A$.

*Page 32.*[31] I have employed in Lansquenet & for Quinquenove your series $a + pa + ppa + p^3a+$ &c. under the expressions of $b$ & of $q$, & I have given very precisely the sums of these series, *pag.* 35, 51 & 112.[32]

Trieze

*Page 59*[33] I am very comfortable that you approve the series

$$\frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \text{\&c.}$$

I have found well some curious things on this matter. I have found, for example, that the advantage of the one who holds the cards on the wager of the players which I call $A$, is

$$\frac{69056823787189897}{241347817621535625}A.$$

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[30]See Page 97.
[31]See pages 110 & 176.
[32]See page 135.
[33]See page 135.

I would make you part of my method, if I did not fear to be too long, I humor myself that it would be to your taste.

*Page 62*[34] It is true that there is an error in this place; however I excuse myself this inattention, & I prefer to have faltered in this place which is simple than in the essential of some method, that which I would not excuse so easily. I thank you for having warned me of it, & I will correct myself in the new edition. I have calculated the following case for four cards, & I have found that $A$ expressing the money of the game, the strength of the one who holds the cards is

$$\frac{56908325}{75285923}A.$$

*Page 80.*[35] The properties of figurate numbers are infinite, I have found some new and singular ones of which I will make you part in the followup to this Letter. That of which you speak to me that the perpendicular bands express the coefficients of the powers, is one of the most beautiful and useful.

Huygens

*Pages* 158 & 159.[36] I am led enough to believe as you that the sense of the Problem of Mr. HUYGENS is rather the one that you give to it than the one of my solution: I have found as you these three numbers 77, 53, 35.

*Page* 137.[37] Your formulas for the cases so much determined as indeterminate are very just, & the remark that you make that that of the determined is applied to the determination of the coefficient of any term is very subtle: my formula for the determined cases is this here:

$$\underset{b}{\overset{p}{\square}} \times \underset{c}{\overset{p-b}{\square}} \times \underset{d}{\overset{p-b-c}{\square}} \times \underset{e}{\overset{p-b-c-d}{\square}} \times \underset{f}{\overset{p-b-c-d-e}{\square}} \quad \text{\&c.}$$

I have one of them also for the indeterminates a little simpler, although the same, as that which is in my Book: Here it is:

$$\underset{B}{\overset{q}{\square}} \times \underset{C}{\overset{q-B}{\square}} \times \underset{D}{\overset{q-B-C}{\square}} \times \underset{E}{\overset{q-B-C-D}{\square}} \times \text{\&c.}$$

$$\times \underset{b}{\overset{p}{\square}} \times \underset{c}{\overset{p-b}{\square}} \times \underset{d}{\overset{p-b-c}{\square}} \times \underset{e}{\overset{p-b-c-d}{\square}} \times \text{\&c.}$$

the whole divided by $q^p$. I understand by $q$ the number of faces of the dice, it is 6 in the ordinary dice.

*Pages* 159 & 160.[38] I do not believe that Mr. HUYGENS has wished understood *minimum*, as you say, after these words: *Inter quos tres albi erunt*; in each case I have wished to divine it, it had been also easy to me in one fashion than the other.

*Page* 162.[39] This observation is very important, & I am very obliged to you having communicated it to me. You have perfectly reasoned, & I am wrong: Here is the manner by which I serve myself that it is arrived. I have resolved five or six years ago the five Problems of Mr. HUYGENS; it was only on the occasion of the extract of the Book of

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[34] See page 142.
[35] See page 32.
[36] See page 220.
[37] See pages 42 & 44.
[38] See page 221.
[39] See page 222.

Mr your brother that is found in the Memoirs of the Academy, that I conclude the project of my Book. This Book achieved, I will give in the third part these five Problems without reviewing them. Now resolving them for the first time, I have resolved them with respect to the Latin enunciation; but being on the point of printing them, I did nothing but to translate the text of Mr. HUYGENS; & finding it obscure, & me no longer reviving the solution, I will give to it a sense that squares not at all with the solution. It is true that these two great numbers 282429536481, 244140625 are the 12$^{\text{th}}$ power of 9 & of 5: the remark is excellent, your two solutions are exquisite. I am no more able than you to take the pain of resolving the Problem on the foot of the enunciation of the proposition, I content myself with your solution, the calculation of it would be too long.

*Page* 177.[40] I would have well wished that you had found for three Players a simple & easy method such as one has for two. I have never been able to come to the end of it, the method of combinations & the analytic method are of an insupportable length. Points

*Page* 178.[41] This is a Problem which would be able to excite your curiosity, that the one where the concern is to determine how much the game must endure when one plays by reducing. I have given in the Corollary, *page 184*[42] the solution of the case where one would play in three games of Piquet: I have the general solution of this Problem. I will send it to you if I would believe that it would be pleasing to you. Mister your nephew, who appears capable to me by his ability of the most difficult things, & who above this is young & has perhaps the leisure, ought to seek the solution of it which is assuredly worthy of him. I have found nearly without calculation that the lot of the one who would wager that the game would endure no more than 26 coups, will be $\frac{16607955}{33554432}$, this which shows that he would have a slight disadvantage, & that the one who would wager that the game will not endure more than 28 coups, will be $\frac{70970250}{133432831}$, which being greater than $\frac{1}{2}$, shows that in this case he would have the advantage. I believe also to have found that in twelve games he would have the advantage to wager that the game will end in less than 124, & that he has disadvantage in 122. Duration of play

I would wish well to know if it is given the pain to determine Proposition 31;[43] I have not at all given my method, here is the formula.

Let $p$ be the number of dice, $\boxed{q}$ the figurate number of order $p$ which corresponds to the rank of the proposed point, beginning from the smallest that one is able to bring forth.

$\boxed{q-6}$ the figurate number of the order $p$ which corresponds to the rank less six of the proposed number, commencing from the smallest that one is able to bring forth. $\boxed{q-12}$ the figurate number of the order $p$ which corresponds to the rank less twelve of the proposed number, commencing from the smallest that one is able to bring forth, &c. $\boxed{q} - p \times \boxed{q-6} + \frac{p\times p-1}{1\cdot 2}\times\boxed{q-12} - \frac{p\times p-1\times p-2}{1\cdot 2\cdot 3}\times\boxed{q-18} + \frac{p\times p-1\times p-2\times\cdot p-3}{1\cdot 2\cdot 3\cdot 4}\times\boxed{q-24}$ –&c. will give the sought number.

One of my friends propose to me some time ago to seek by a formula how many coups there are to bring forth precisely the number of 6 or of 5 with a certain number $p$ of dice. I have found for formula $(m-1)^{p-q}$ multiplied by as many products of the quantities $p \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \cdot \frac{p-4}{5}$ &c. as there are units in $q$. I call $m$ the number of faces of the dice, $m$ is 6 in the ordinary dice.

*In the rest of this Letter I will made to Mr. Bernoulli part of my method, in order to find the sum of a sequence of terms which have some constant differences, & I finished it by*

[40] See page 242.

[41] See page 268.

[42] See page 276.

[43] See page 46.

*explaining to him the conditions of the Lottery of Lorraine. As the method is found in this Book at page 63, & the solution of the Problem of the Lottery at page 257, I will not repeat here that which one sees besides.*