SUPPLÉMENT AU MÉMOIRE SUR UN PROBLÉME DU CALCUL DES PROBABILITÉS INSÉRÉ DANS LE VI^{e} VOLUME DES ACTES*

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In giving the Solution of this Problem, resolved by the late Mr. *Jacques Bernoulli*, in his posthumous work *De arte conjectandi*, I have first avowed to have seen only the enunciation and the result, such as they are found in a memoir of Mr. *Mallet*, in the *Acta Helvetica, Vol. VII*. Although my result was sensibly different from the one that Mr. *Bernoulli* must have found, I did not permit myself to suspect his solution, but I believe duty to publish mine which I know to be just, by expecting that the source of this difference is manifested. It was only a bad understanding in the enunciation of the Problem which can explain it; & effectively, having had since the occasion to procure myself the work of the late Mr. *Bernoulli*, I have found enunciated the Problem in the following manner.

> Duo collusores A & B, *tessera in alueum proiecta*, conuentiunt inter se, ut quot eius puncta ceciderint, tot iactus uterque instituat, illeque depostitum ausserat, qui plura summatim puncta iecerit; sin autem aequalis punctorum numerus ambobus contingat, aequaliter etiam depositum inter se partiantur. Mox vero collusorum alter B, ludi peratesus, loco incertae aleae certum punctorum numerum assumere et punctis 12 pro rata sua acquiescere mavult. Annuit A. Quaeritur uter altero et quanto potiorem vincendi spem habeat?[1]

& first after this enunciation Mr. Bernoulli distinguishes two cases, by saying: *Determinandum ante omnia, an primus tesserae iactus accenseri debeat iactibus collusoris A, nec ne?*[2] & he treats both. Now I have considered only the first case & the

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[1]Two players A and B, agree to throw a die, and that each will then have the same number of throws as points thrown, the winner being the one who throws the greatest aggregated number of points. Should they both obtain equal numbers of points, the stake will be divided equally. However B tires of this game and instead of an uncertain number of points, wishes to take a particular number, and indeed wishes to acquire 12 at an appropriate cost. A agrees. It is required to determine for each the strength of their hope of winning.

[2]It must be determined first of all whether the first throw is to be counted as for A or not.

ratio that I found between the expectations of the two players is perfectly in accord with the one of *Bernoulli*, namely A:B= $46529 : 46783$.

Having made so much as to publish my solution, on the idea that it was different from that of Mr. *Bernoulli*, in order to render it complete I wish to add to it here the development of another cas, where the points of the first cast are not counted, this which, as one sees easily, renders the share of player B still more advantageous, although, to judge it according to the mistaken reasoning that I have alleged in my first memoir as an example of the circumspection which it is necessary to bring in these sorts of questions, player A has equally $12\frac{1}{4}$ points against the 12 points of B. For

if he casts 1, he has 0 points &	$\frac{7}{2}$ of expectation	for 1 following
2	$\frac{14}{2}$	2
3	$\frac{21}{2}$	3
4	$\frac{28}{2}$	4
5	$\frac{35}{2}$	5
6	$\frac{42}{2}$	6
	$\frac{1}{6}\cdot\frac{147}{2} = 12\frac{1}{4}$	

all as in the first case.

Before beginning the enumeration of the expectations of player A, it is necessary to make that of all the cases which can take place, when with any number whatever of dice one wishes to bring forth a certain number of points. Let for this effect the number of dice be $= n$ the number of the points $= \mu$ & the number of cases $= \Lambda$, I have shown in the first memoir, that there will be

$$\Lambda = \left(\frac{\mu-1}{n-1}\right) - \left(\frac{n}{1}\right)\left(\frac{\mu-7}{n-1}\right) + \left(\frac{n}{2}\right)\left(\frac{\mu-13}{n-1}\right) - \left(\frac{n}{3}\right)\left(\frac{\mu-19}{n-1}\right) + \text{\&c.}$$

where the characters $\left(\frac{n}{1}\right)$, $\left(\frac{n}{2}\right)$, $\left(\frac{n}{3}\right)$, &c. are the coefficients of the powers of the developed binomial, & $\left(\frac{\mu-1}{n-1}\right)$, $\left(\frac{\mu-7}{n-1}\right)$, $\left(\frac{\mu-13}{n-1}\right)$, &c. some similar products, namely

$$\left(\frac{\mu-1}{n-1}\right) = \frac{(\mu-1)(\mu-2)(\mu-3)(\mu-4)\cdots(\mu-n+1)}{(n-1)(n-2)(n-3)\cdots 1}$$

$$\left(\frac{\mu-7}{n-1}\right) = \frac{(\mu-7)(\mu-8)(\mu-9)(\mu-10)\cdots(\mu-n+7)}{(n-1)(n-2)(n-3)\cdots 1}$$

&c. &c.

whence it will be easy to construct the following small table, extended to $n = 6$ &

$\mu = 12$.

n	2	3	4	5	6
μ	Λ	Λ	Λ	Λ	Λ
2	1				
3	2	1			
4	3	3	1		
5	4	6	4	1	
6	5	10	10	5	1
7	6	15	20	15	6
8	5	21	35	35	21
9	4	25	56	70	56
10	3	27	80	126	126
11	2	27	104	205	252
12	1	25	125	305	456
	36	160	435	762	918

Let $2M$ be the deposited sum that A wins in bringing forth by all his casts beyond twelve points, while he wins only the half M, in bringing forth twelve points only; let moreover the expectations of the player A, be $a,\ b,\ c,\ d,\ e,\ f$, in bringing forth on the first cast Ace, Deuce, Three, Four, &c. & their sum

$$a + b + c + d + e + f = s,$$

the entire expectation of A will be $\frac{1}{2}s$, & that of B will be $2M - \frac{1}{2}s$.

This remarked, if player A casts the ace at the first cast, he will have to recast the die only one single time, & being able to expect from this cast only six points at most, he has lost & his expectation will be $a = 0$.

If player A casts the Deuce at the first cast, he can recast the die two times in sequence, or else he will cast two dice all at once. With these two dice he could expect only twelve points at most, & there is only one way to bring them forth. Thus, the number of all possible cases to cast two dice being 6^2, the probability that player A has to cast 12 points is $\frac{1}{6^2}$, & hence his expectation on the half of the gain $b = \frac{1}{36}M$.

If A casts the Three, he will take three dice, & he will have 25 cases which can give 12 points, & $6^3 - 130 = 56$ which can produce beyond 12. The probability is $= \frac{25}{6^3}$ for the first case and $\frac{56}{6^3}$ for the other. In this one here A wins all, in the first only the half. His expectation is therefore

$$c = \frac{25}{6^3}M + \frac{2.56}{6^3}M = \frac{137}{6^3}M.$$

If A casts the Four, he will have the second cast with four dice, where there are 125 different ways to bring forth 12 points & $6^4 - 435 = 861$ ways to bring forth beyond 12. The probability of the first is $\frac{125}{6^4}$, of the other it is $\frac{861}{6^4}$ & the sum of the expectations

$$d = \frac{125}{6^4}M + \frac{2.861}{6^4}M = \frac{1847}{6^4}M.$$

If the first cast gives five points the second itself will be with five dice, & because there are 305 cases which produce 12 points, & $6^5 - 762 = 7014$ cases which give advantage, the expectation of player A will be

$$e = \frac{305 + 2.7014}{6^5}M = \frac{14333}{6^5}M.$$

Finally if the first cast authorizes him to take six dice for the second cast, the expectation of A for the half of the gain M will be $\frac{456}{6^6}M$, & for the entire gain $2M$ it will be $(6^6 - 918).2M$, or else $\frac{91476}{6^6}M$, & consequently

$$f = \frac{91932}{6^6}M = \frac{15322}{6^5}M.$$

The sum of these expectations, reduced to the same denominator will be therefore

$$s = \frac{6^3 + 6^2.137 + 6.1847 + 14333 + 15322}{6^5}M$$

& hence the entire expectation of player

$$A = \frac{1}{6}s = \frac{45885}{6^6}M$$

& that of player

$$B = 2M - \frac{1}{6}s = \frac{47427}{6^6}M.$$

Thus the expectation of A is to that of B as

$$45885 : 47427 = 15295 : 15809.$$

A ratio that Mr. *Bernoulli* has also found.