Solution of a very difficult Question in the Calculus of Probabilities

Leonhard Euler*

E412
Mémoires de l'académie de Berlin [25] (1769), 1771, p. 285–302

1. This is the plan of a lottery which has furnished me this question, that I propose to develop. This lottery had five classes, each of 10000 tickets, among which there were 1000 prizes in each class and consequently 9000 blanks. Each ticket must pass through all five classes; and this lottery had it in particular that beyond the prizes of each class one was committed to pay a ducat to each of those for which the tickets had passed through all five classes without winning anything. One sees well that this last expense to which the lottery is committed is very uncertain, seeing that it would be possible, on the one hand, that all the prizes in each class fell on the same numbers, and in this case there would be 9000 to each of which there must be a ducat. Now, on the other hand, if all the prizes of five classes fell on different numbers, there would be in all 5000 winning tickets, and as many losing, so that in this case the settled expense would amount to only 5000 ducats. The one and the other of these two cases being nearly impossible, the question is to determine the number of ducats that the lottery will probably be obliged to pay. For this effect, it is necessary to make a perfect denumeration of all the possible cases, for each number of those which lose in all five classes, for the least of 5000 up to the greatest of 9000.

2. In order to render this research both more general and more bright, I will put

1 ˚.	The number of the classes in the lottery	$= k.$
2 ˚.	The number of prizes in each class	$= n.$
3 ˚.	The number of blank tickets in each	$= m.$
4 ˚.	Therefore the number of all the tickets	$= m + n.$

Each of these $m + n$ tickets passes through all k classes, in each of which it will win or will lose; and if it happens that it does not win in all the classes, then it will enjoy the mentioned benefit of one ducat. Therefore there is concern to estimate, according to the rules of probability, the number of tickets which will pass through all the classes without winning anything; and first, in order to know the limits of this number, we suppose that all the prizes in each class fall on the same tickets; in this case therefore, there will be only n tickets which win, and all the others, of which the number is $= m$,

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. January 1, 2022 will be in the case of receiving one ducat, so that this expense is of m ducats to the fund of the lottery, and it is the greatest possible. Now, it will be least when it will happen that all the prizes of each class fall on different tickets; in this case, the number of those which win, in any class that this be, will be $= kn$, and therefore, the number of those who lose

$$= m + n - kn = m - (k-1)n.$$

Consequently, the expense in this case will be only of $m-(k-1)n$ ducats, in supposing that the number m is greater than $(k-1)n$; because if it were equal to it, or even smaller, this expense would reduce to nothing.

3. Here is therefore the question of which it is necessary to seek the solution. It is concerned with finding, among all the possible cases, those where the number of those who lose in all k classes will be either m or $m-1$ or $m-2$ or $m-3$ or $m-4$ etc. up to $m-(k-1)n$. Next one knows, by the rules of probability, that each of these numbers divided by the number of all the possible cases expresses the probability that this case exist, which will be accordingly greater as it approaches nearer to unity; and if it became equal to unity, this would be an indication of one entire certitude. This happens in the case where one single class, where $k = 1$, expecting that the number of losers is then certainly $= m$, and the expression for the probability becomes then $= 1$, or else it marks an entire certitude.

But, if the lottery is composed of many classes, such that $k > 1$, one will have always many cases to develop, according as the number of those who lose in the classes is either m or $m-1$ or $m-2$ or $m-3$ etc. up to $m-(k-1)n$; and having found the probability of each of these cases, since it must be by all necessity that any one of them exist, it is evident that the sum of all these probabilities together is equal to unity or to the measure of an entire certainty. This property serves besides to verify the solutions that one gives to the similar questions; but here it will serve me to find the same solution to the proposed problem, and I doubt strongly that without this help one can succeed there.

4. I suppose first that one has drawn already the first class and that the prizes have fallen on the tickets marked A, B, C, D, E etc., of which the number is $= n$. Now, in passing to the second class, where there are again n prizes, the number of all the tickets being $= m + n$, I remark that the number of all the possible variations among the n tickets to which the prize is attached, without having regard to their order, is

$$= \frac{(m+n)(m+n-1)(m+n-2)\cdots(m+1)}{1 \cdot 2 \cdot 3 \cdots n}$$

and if one wishes also to have regard to the diversity of the order following which they are drawn out successively, one has only to omit the denominator, and the number of all the possible cases will be

$$= (m+n)(m+n-1)(m+n-2)\cdots(m+1).$$

Now, considering also the diversity of the order, the number of all the cases where the prize meets with the same tickets A, B, C, D etc. which were drawn from the first class and of which the number is $= n$, it is expressed thus

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots n.$$

Therefore, in order that the prizes of the second class fall on the same tickets as in the first, the probability is

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{(m+1)(m+2)(m+3)\cdots(m+n)},$$

and that the same thing occurs also in the third, the probability is equal to the square of this expression, in the fourth to the cube, and thus in order. Consequently, that in all k classes the prizes fall on the same tickets, the probability will be

$$= \left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{(m+1)(m+2)(m+3)\cdots(m+n)}\right)^{k-1}.$$

5. I remark on this expression, 1 ° . that the number of all the possible variations in relation to the tickets which are met with a prize, in all the classes together, is

$$= ((m+n)(m+2)(m+3)\cdots(m+n))^{k-1},$$

in keeping also count of the diversity in the order where the winning tickets are drawn successively; next 2 ° . that the number of all the possible cases that precisely the tickets marked A, B, C, D etc. are met with the prizes in all the classes is

$$(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)^{k-1},$$

so that this number divided by the former expresses the probability that this case exists, as I come to find it.

But the most essential remark, which will lead me to the proposed goal, consists in this that the number of all the possible cases where, in all k classes, the prizes are met with the same tickets marked A, B, C etc. depends uniquely 1 ° . on the number of prizes n, or else on those of the tickets A, B, C, D etc. which won in the first class, and 2 ° . on the number of classes k of the lottery; so that the number of other tickets, which is $= m$, do not enter at all into consideration; or else, however great be the number of all the tickets, the number of cases which make the same tickets win in all the classes remains always the same. Let there be no doubt that I speak here always of all the possible variations, as many in the same tickets as in their order.

6. We put for the sake of brevity

$$((m+1)(m+2)(m+3)\cdots(m+n))^{k-1} = M$$

and

$$(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)^{k-1} = \alpha;$$

and the number of cases where the number of those who lose in all the classes let it $= m$ will be

$$= \alpha;$$

and the probability that some one of the cases exists will be $= \frac{\alpha}{M}$.

Now, I pass to the second case, where the number of those which lose in all the classes is $= m - 1$; and I remark that beyond the tickets marked A, B, C, D etc.

which won in the first class, it is necessary that one of the others, of which the number is $= m$, wins also in one or many of the other classes; since this good fortune is able to happen to each of the m tickets, the number of all these cases will be expressed by

$$\beta m,$$

where β no longer contains the number m, but depend uniquely on the combinations with the other tickets which win in the remaining classes.

In the same way for the third case, where the number of those which lose in all classes is $= m - 2$, it is necessary to combine two tickets of those which lost in the first; which accommodate $m(m-1)$ variations, the number of all these cases will have this form

$$\gamma m(m-1).$$

For the fourth case, where the number of losers through all the classes is $m-3$, the number of all the possible cases will be

$$= \delta m(m-1)(m-2);$$

and thus so forth for the remaining cases, where the number of losers in all the classes is either $m-4$ or $m-5$ or $m-6$ etc. up to $m-(k-1)n$.

7. In order to see at a glance all these suppositions, I will represent them in this fashion:

Number of those who lose in all the classes	Number of all the cases where this occurs	Probability that any one of these cases exists
m	α	$\frac{\alpha}{M}$
$m-1$	βm	$\frac{\beta m}{M}$
$m-2$	$\gamma m(m-1)$	$\frac{\gamma m(m-1)}{M}$
$m-3$	$\delta m(m-1)(m-2)$	$\frac{\delta m(m-1)(m-2)}{M}$
$m-4$	$\epsilon m(m-1)(m-2)(m-3)$	$\frac{\epsilon m(m-1)(m-2)(m-3)}{M}$
$\vdots$	$\vdots$	$\vdots$
$m-(k-1)n$	$\omega m(m-1)\cdots(m-kn+n+1)$	$\frac{\omega m(m-1)\cdots(m-kn+n+1)}{M}$

where, for the sake of brevity, I have put

$$M = ((m+1)(m+2)(m+3)\cdots(m+n))^{k-1}.$$

Having already found the first value

$$\alpha = (1\cdot 2\cdot 3\cdot 4\cdots n)^{k-1},$$

all reverts to seeking the values of the following letters $\beta, \gamma, \delta, \epsilon$ etc., that which could be done following the principles of combinations and variation; but this would demand some very thorny and tedious researches, which one would have even much difficulty to motivate if remotely one could discover the law of the progression; again such a law concluded only by induction would be strongly subject to caution.

8. But the consideration that all these probabilities together must be equal to unity furnishes us a quite easy route in order to determine all these quantities $\alpha, \beta, \gamma, \delta$ etc. We have only to satisfy this equation:

$$M = \alpha + \beta m + \gamma m(m-1) + \delta m(m-1)(m-2) + \epsilon m(m-1)(m-2)(m-3) + \text{etc.},$$

in observing that the quantities $\alpha, \beta, \gamma, \delta, \epsilon$ etc. do not depend on the number m, but that they are uniquely determined by the two others, n and k.

Here is in what manner one must conduct the reasoning in order to arrive at this goal. Since this equation must always take place, whatever value one gives to the number m, we put first $m = 0$; and we will have

$$(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)^{k-1} = \alpha,$$

whence we obtain the same value for α that I have assigned before. Next we put for m successively the numbers 1, 2, 3, 4 etc. in order to have these equations

$$\begin{aligned}
&(2 \cdot 3 \cdot 4 \cdots (n+1))^{k-1} = \alpha + \beta, \\
&(3 \cdot 4 \cdot 5 \cdots (n+2))^{k-1} = \alpha + 2\beta + 2\gamma, \\
&(4 \cdot 5 \cdot 6 \cdots (n+3))^{k-1} = \alpha + 3\beta + 6\gamma + 6\delta, \\
&(5 \cdot 6 \cdot 7 \cdots (n+4))^{k-1} = \alpha + 4\beta + 12\gamma + 24\delta + 24\epsilon, \\
&(6 \cdot 7 \cdot 8 \cdots (n+5))^{k-1} = \alpha + 5\beta + 20\gamma + 60\delta + 120\epsilon + 120\zeta, \\
&\text{etc.,}
\end{aligned}$$

whence one will draw without difficulty successively the values of all the letters $\beta, \gamma, \delta, \epsilon, \zeta$ etc. up to the last ω, which one will find easily to be $= 1$, since the number of cases where all the prizes fall on the different tickets is

$$m(m-1)(m-2)(m-3)\cdots(m-(k-1)n+1).$$

9. Let, for the sake of brevity, the value of M, in setting in general $m = \lambda$, be indicated in this way

$$M^{(\lambda)};$$

and we will have

$$\begin{aligned}
M^{(0)} &= \alpha, \\
M^{(1)} &= \alpha + \beta, \\
M^{(2)} &= \alpha + 2\beta + 2\gamma, \\
M^{(3)} &= \alpha + 3\beta + 6\gamma + 6\delta, \\
M^{(4)} &= \alpha + 4\beta + 12\gamma + 24\delta + 24\epsilon, \\
&\text{etc.}
\end{aligned}$$

whence, taking the differences,

$$\begin{aligned}
M^{(1)} - M^{(0)} &= \beta, \\
M^{(2)} - M^{(1)} &= \beta + 2\gamma, \\
M^{(3)} - M^{(2)} &= \beta + 4\gamma + 6\delta, \\
M^{(4)} - M^{(3)} &= \beta + 6\gamma + 18\delta + 24\epsilon, \\
&\text{etc.;}
\end{aligned}$$

and the second differences will be

$$\begin{aligned}
M^{(2)} - 2M^{(1)} + M^{(0)} &= 2\gamma, \\
M^{(3)} - 2M^{(2)} + M^{(1)} &= 2\gamma + 6\delta, \\
M^{(4)} - 2M^{(3)} + M^{(2)} &= 2\gamma + 12\delta + 24\epsilon, \\
&\text{etc.;}
\end{aligned}$$

and more, the third differences

$$\begin{aligned}
M^{(3)} - 3M^{(2)} + 3M^{(1)} - M^{(0)} &= 6\delta, \\
M^{(4)} - 3M^{(3)} + 3M^{(2)} - M^{(1)} &= 6\delta + 24\epsilon, \\
&\text{etc.;}
\end{aligned}$$

and the fourth

$$\begin{aligned}
M^{(4)} - 4M^{(3)} + 6M^{(2)} - 4M^{(1)} + M^{(0)} &= 24\epsilon, \\
&\text{etc.;}
\end{aligned}$$

After the continuation of these differences, one would have no doubt.

10. All these values $M^{(0)}, M^{(1)}, M^{(2)}$ etc., derived from

$$M = ((m+1)(m+2)(m+3)\cdots(m+n))^{k-1},$$

being known and independent of the number m, the quantities $\alpha, \beta, \gamma, \delta$ etc., which comprise the solution to our question, will be determined thus:

$$\begin{aligned}
\alpha &= M^{(0)}, \\
\beta &= \frac{M^{(1)} - M^{(0)}}{1}, \\
\gamma &= \frac{M^{(2)} - 2M^{(1)} + M^{(0)}}{1 \cdot 2} \\
\delta &= \frac{M^{(3)} - 3M^{(2)} + 3M^{(1)} - M^{(0)}}{1 \cdot 2 \cdot 3}, \\
\epsilon &= \frac{M^{(4)} - 4M^{(3)} + 6M^{(2)} - 4M^{(1)} + M^{(0)}}{1 \cdot 2 \cdot 3 \cdot 4} \\
&\text{etc.}
\end{aligned}$$

Now, among these diverse values derived from M, we know the following relations

$$M^{(1)} = \left(\frac{n+1}{1}\right)^{k-1} M^{(0)},$$
$$M^{(2)} = \left(\frac{n+2}{2}\right)^{k-1} M^{(1)},$$
$$M^{(3)} = \left(\frac{n+3}{3}\right)^{k-1} M^{(2)},$$
$$M^{(4)} = \left(\frac{n+4}{4}\right)^{k-1} M^{(3)},$$
etc.,

the first being
$$M^{(0)} = (1\cdot 2\cdot 3\cdot 4\cdots n)^{k-1}.$$
Whence, by this single value $M^{(0)}$, we will have

$$\beta = \tfrac{\alpha}{1}\left(\left(\tfrac{n+1}{1}\right)^{k-1} - 1\right),$$
$$\gamma = \tfrac{\alpha}{1\cdot 2}\left(\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\right)^{k-1} - 2\left(\tfrac{n+1}{1}\right)^{k-1} + 1\right),$$
$$\delta = \tfrac{\alpha}{1\cdot 2\cdot 3}\left(\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\cdot\tfrac{n+3}{3}\right)^{k-1} - 3\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\right)^{k-1} + 3\left(\tfrac{n+1}{1}\right)^{k-1} - 1\right),$$
$$\epsilon = \tfrac{\alpha}{1\cdot 2\cdot 3\cdot 4}\left\{\begin{array}{l}\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\cdot\tfrac{n+3}{3}\cdot\tfrac{n+4}{4}\right)^{k-1} - 4\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\cdot\tfrac{n+3}{3}\right)^{k-1}\\ +6\left(\tfrac{n+1}{1}\cdot\tfrac{n+2}{2}\right)^{k-1} - 4\left(\tfrac{n+1}{1}\right)^{k-1} + 1\end{array}\right\}$$
etc.

of which the progression is equally evident.

11. In order to better see the nature of these numbers $\alpha, \beta, \gamma, \delta$ etc., we develop some particular cases; and we suppose first that there is only a single prize in each class, so that $n = 1$, the number of classes remaining $= k$. Let $k = \pi + 1$, in order to have $k - 1 = \pi$. The number of all tickets in each class will be therefore $= m + 1$ and $M = (m+1)^\pi$, and therefore

$$M^{(0)} = 1, \quad M^{(1)} = 2^\pi, \quad M^{(2)} = 3^\pi, \quad M^{(3)} = 4^\pi, \quad M^{(4)} = 5^\pi, \quad \text{etc.},$$

whence we draw the following values:

	if $\pi = 1$	if $\pi = 2$	if $\pi = 3$	if $\pi = 4$	if $\pi = 5$
$\alpha=1$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$
$\beta=\frac{2^\pi-1}{1}$	$\beta = 1$	$\beta = 3$	$\beta = 7$	$\beta = 15$	$\beta = 31$
$\gamma=\frac{3^\pi-2\cdot 2^\pi+1}{1\cdot 2}$	$\gamma = 0$	$\gamma = 1$	$\gamma = 6$	$\gamma = 25$	$\gamma = 90$
$\delta=\frac{4^\pi-3\cdot 3^\pi+3\cdot 2^\pi-1}{1\cdot 2\cdot 3}$	$\delta = 0$	$\delta = 0$	$\delta = 1$	$\delta = 10$	$\delta = 65$
$\epsilon=\frac{5^\pi-4\cdot 4^\pi+6\cdot 3^\pi-4\cdot 2^\pi+1}{1\cdot 2\cdot 3\cdot 4}$	$\epsilon = 0$	$\epsilon = 0$	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 15$
etc.	$\zeta = 0$	$\zeta = 0$	$\zeta = 0$	$\zeta = 0$	$\zeta = 1$

12. Now let the number of prizes in each class be $n = 2$, the two other numbers m and $k = \pi + 1$ remaining indeterminant. Therefore, since

$$M = (m + 1)^\pi (m + 2)^\pi,$$

we will have

$$M^{(0)} = 2^\pi, \quad M^{(1)} = 6^\pi, \quad M^{(2)} = 12^\pi, \quad M^{(3)} = 20^\pi \quad \text{etc.},$$

and therefore:

	if $\pi = 1$	if $\pi = 2$	if $\pi = 3$
$\alpha = 1$	$\alpha = 2$	$\alpha = 4$	$\alpha = 8$
$\beta = \frac{6^\pi - 2^\pi}{1}$	$\beta = 4$	$\beta = 32$	$\beta = 208$
$\gamma = \frac{12^\pi - 2 \cdot 6^\pi + 2^\pi}{1 \cdot 2}$	$\gamma = 1$	$\gamma = 38$	$\gamma = 652$
$\delta = \frac{20^\pi - 3 \cdot 12^\pi + 3 \cdot 6^\pi - 2^\pi}{1 \cdot 2 \cdot 3}$	$\delta = 0$	$\delta = 12$	$\delta = 576$
etc.	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 188$
	$\zeta = 0$	$\zeta = 0$	$\zeta = 24$
	$\eta = 0$	$\eta = 0$	$\eta = 1$

13. It would be useless to develop many cases, since the determination of the numbers $\alpha, \beta, \gamma, \delta$ etc. would demand some too difficult calculations, which itself, at the end of the computations, would furnish us nothing enlightening on the question of which there was concern. Whence one understands that, if one wanted to apply these formulas to the example of the lottery reported as the beginning, in supposing

$$n = 1000, \quad m = 9000 \quad \text{and} \quad k = 5,$$

whence would result the number

$$M = (9001 \cdot 9002 \cdot 9003 \cdots 10000)^4,$$

and those which are derived from it

$$M^{(0)} = (1 \cdot 2 \cdot 3 \cdots 1000)^4$$
$$M^{(1)} = (2 \cdot 3 \cdot 4 \cdots 1001)^4$$
$$M^{(2)} = (3 \cdot 4 \cdot 5 \cdots 1002)^4$$
$$M^{(3)} = (4 \cdot 5 \cdot 6 \cdots 1003)^4$$
$$\text{etc.}$$

one would be obliged to plunge oneself into the terrible calculations before arriving at the knowledge of the numbers $\alpha, \beta, \gamma, \delta$ etc., of which the number climbs to 4001. Next, it would be necessary still to multiply each of these numbers by the coefficients assigned in §7, in order to have the numbers of all the cases where each variety is able to occur. And finally, having found all these numbers, there would remain to divide each one by the number M, in order to have the probability that the number of those who lose in all the classes be either 9000 or 8999 or 8998, until one arrives at 5000.

14. It is quite certain that a person will never undertake this immense labor, in the sole view to respond to the entrepreneurs of the mentioned lottery, to how much they must probably estimate the expenditure to which they commit in promising a ducat to each of those who won't have won anything in all 5 classes. Therefore, if there is not another means to satisfy this question, one would be well obliged to regard the solution as morally impossible, and there would be no other choice to make than to advise to the entrepreneurs of a similar lottery to hold themselves in it to some number midway between the greatest sum of 9000 ducats and the least sum of 5000 ducats, which constitute the limits of this expense. In the remainder, if it is agreed only to draw this lottery one time, it would not be worth even the difficulty to commit to this work, when even it would not be so difficult, since a single event never settles the probability. But if one wanted to repeat many times in sequence this same lottery, the question would become more important, since then the said expense would be sometimes greater, sometimes less; and it is only in this case that one could be assured that the mean among all these expenses will approach more nearly the sum determined by the laws of probability, the more times one will repeat the drawing of this same lottery. Therefore it is this mean sum that the laws of probability must reveal to us.

15. Now, no matter the insurmountables that seem in the calculations in order to find this sum, there is encountered a certain fortunate circumstance which renders extremely easy the execution of all the calculations, so that one has no need even to calculate the values of the numbers $\alpha, \beta, \gamma, \delta$ etc. One has only to take to the general formulas given in §7; and since for each number of ducats to which the expense is able to arise, the probability is as follows:

that the expense be as many ducats	the probability is
m	$\frac{\alpha}{M}$
$m-1$	$\frac{\beta m}{M}$
$m-2$	$\frac{\gamma m(m-1)}{M}$
$m-3$	$\frac{\delta m(m-1)(m-2)}{M}$
$\vdots$	$\vdots$
$m-(k-1)n$	$\frac{\omega m(m-1)(m-2)\cdots(m-(k-1)n+1)}{M}$,

the sum of each expense multiplied by the probability will give the true mean expense that we seek, which will be by consequence

$$= \frac{\alpha m + \beta m(m-1) + \gamma m(m-1)(m-2)\cdots \omega m(m-1)\cdots(m-(k-1)n)}{M},$$

and I remark that the value of this expression is able to be assigned without one having any need to develop either the numbers $\alpha, \beta, \gamma, \delta$ etc. or likewise the denominator M; which is without doubt an event which one could not do.

16. Having made to see hereafter that the numbers $\alpha, \beta, \gamma, \delta$ etc. do not depend on the number m and that they must be such that there be

$$\begin{aligned}\alpha &+ \beta m + \gamma m(m-1) + \delta m(m-a)(m-2) + \cdots \\ &+ \omega m(m-1)(m-2)\cdots(m-(k-1)n+1) \\ &= M = ((m+1)(m+2)\cdots(m+n))^{k-1},\end{aligned}$$

it follows first that writing $m-1$ instead of m, it is necessary that there be

$$\begin{aligned}&\alpha+\beta(m-1)+\gamma(m-1)(m-2)+\cdots+\omega(m-1)(m-2)\cdots(m-(k-1)n)\\&=(m(m+1)(m+2)\cdots(m+n-1))^{k-1},\end{aligned}$$

the numbers $\alpha, \beta, \gamma, \delta$ etc. being the same as above. But this last expression

$$\alpha+\beta(m-1)+\gamma(m-1)(m-2)+\text{etc.},$$

being multiplied by m, gives precisely the numerator of the fraction that we come to assign to the probable amount of the expense; whence we conclude this expense

$$=\frac{m(m(m+1)(m+2)\cdots(m+n-1))^{k-1}}{((m+1)(m+2)\cdots(m+n))^{k-1}},$$

which evidently is reduced to this

$$=m\left(\frac{m}{m+n}\right)^{k-1},$$

of which the application is easily made to each case proposed, but for which one would need to calculate neither the values of the letters $\alpha, \beta, \gamma, \delta$ etc. nor the number M.

Therefore here is, against all expectation, a solution as simple as beautiful of our question, by which we know that in general, the number of the classes being k, the number of prizes of each class $= n$ and the number of all tickets $= m+n$, the expense in question must be estimated

$$=m\left(\frac{m}{m+1}\right)^{k-1}.$$

17. For the case of the lottery described at the beginning, where $k=5, n=1000$ and $m=9000$, the expense in favor of those who win nothing in all five classes must be estimated at $9000\left(\frac{9}{10}\right)^4$ ducats, which makes $5904\frac{9}{10}$ ducats, whence one sees that the mean is much closer to the lowest limit 5000 than the greatest 9000.

Let, in order to give another example, the number of classes be again $k=5$, the number of prizes in each class $n=8000$ and that of all the tickets $m+n=50000$; therefore $m=42000$; and when one is committed to pay also a ducat to each of those who pass through the five classes without winning anything, this expense must be estimated, according to the rules of probability, at $42000\left(\frac{42}{50}\right)^4$, that is to say at $20910\frac{6}{10}$ ducats.

18. In general, I remark on the estimate of this expense that I come to find $= m\left(\frac{m}{m+n}\right)^{k-1}$ that, when there would be a single class, it will be

$$=m,$$

to which case the probability becomes one entire certitude.

But if the lottery is composed of two classes, this expense is

$$=\frac{mm}{m+n};$$

for three classes it becomes

$$= \frac{m^3}{(m+n)^2},$$

for four

$$= \frac{m^4}{(m+n)^3},$$

and so forth; so that it declines in ratio of $m + n$ to m for each class more. Therefore, if the number of classes were infinite, this expense would reduce to nothing, no matter how small be the number of prizes in ratio to all the tickets.

As this simple formula comes to be concluded from an extremely difficult calculation, there is no doubt that there is another method, very simple, which leads directly to it without any detour. In effect, the sole consideration of this formula furnishes us first the reasonings that are necessary to make in order to attain it, that I proceed to set all in their time.

19. One has only to examine successively all the classes, in reflecting that each class contains in all $m + n$ tickets, among which there are n winners and m losers. Therefore, the first class being drawn, there will be certainly m tickets which will be losers; those entering into the second class, it is probable that they will be among the ones which win, and this in the ratio of the number of all the tickets $m + n$ to number of prizes n; therefore, of these m tickets which have lost in the first class, there will be probably $m \cdot \frac{m}{m+n}$ who will win in the second class; and therefore, the number of those who pass through the first two classes without winning anything must be estimated

$$= m \cdot \frac{m}{m+n}.$$

Now, those tickets enter into the third class; and, by the same reason, their entire number will be to that of the tickets which will lose also in this class as $m + n$ to m; consequently, the number of tickets which will pass through the first three classes without winning anything will be probably

$$= \left(\frac{m}{m+n}\right)^2.$$

By this same reasoning, one finds that the number of tickets which will pass probably through four classes without winning anything will be

$$= \left(\frac{m}{m+n}\right)^3;$$

and in general, if the number of classes is $= k$, the number of tickets which will pass through all these classes without winning anything must be fixed, according to the rules of probability, at

$$= m\left(\frac{m}{m+n}\right)^{k-1};$$

and if one commits to pay to each a ducat, this expense must be estimated at $m\left(\frac{m}{m+n}\right)^{k-1}$ ducats, which is precisely the sum that I have found above.

20. If this route is preferable to the first because of its simplicity, the first has some very considerable advantages, in revealing to us in detail the probability that the expense equals precisely a given sum. Because, as it is not even probable that the expense actually is the same as the probability indicates, it is very important that the denumeration of all the possible cases be well known to us, in order to put us in a state to judge the probability of each. But the last method has however this advantage over the first, that it is able to be applied to some cases where the classes of the lottery do not contain the same number of prizes; which circumstance would render nearly impossible the first method. However, it is necessary always to propose that the number of all tickets be the same in all the classes, since, without this condition, the question of which it is concerned could not be put.

Therefore let l be the number of all tickets in each class, and we put the number of those who lose: in the first class $= m$, in the second $= m'$, in the third $= m''$, in the fourth $= m'''$, and so on. This put, the number of tickets which will lose in all classes will be probably

$$m \cdot \frac{m'}{l} \cdot \frac{m''}{l} \cdot \frac{m'''}{l} \cdot \frac{m^{iv}}{l} \cdot \frac{m^{v}}{l} \cdot \text{etc.},$$

until one has run through all the classes. Whence one sees that if there was a single class where all the tickets won, some one of the numbers m, m', m'', m''' etc. would vanish and the number found would be reduced to zero, this which would be no more the measure of the probability, but a complete certitude.

21. In order to give an example, we suppose that there is a lottery composed of 5 classes, each one containing 10000 tickets, and of which the first contains 1000 prizes, the second 2000, the third 3000, the fourth 4000 and the fifth 5000. We will have therefore $l = 10000$; and the numbers of tickets which lose in each class will be

$$m = 9000,\ m' = 8000,\ m'' = 7000,\ m''' = 6000,\ m^{iv} = 5000.$$

And therefore, the number of tickets which will pass through all 5 classes without winning anything will be, consistently to the rules of probability,

$$= 10000 \cdot \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} \cdot \frac{6}{10} \cdot \frac{5}{10} = 1512,$$ [1]

or else, one is able to estimate that there will be only 1512 tickets which will lose in all 5 classes; therefore, it is probable that of all 10000 tickets there will be 8488 which will draw some prize in one or more classes. Consequently, for those who would be interested in this lottery, one is able to say that the probability is $\frac{8488}{10000}$ or $\frac{1061}{1250}$, that they will not pass through all 5 classes without winning anything.

[1] The original edition carries, by error, $= 9000 \cdot \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} \cdot \frac{6}{10} \cdot \frac{5}{10} = 1512$. L. G. D.