VII. ON THE DURATION OF LIFE

JEAN D'ALEMBERT
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1. The reflections which I have just proposed to you on the ordinary analysis of chances, will lead me to some others on the manner in which we calculate the probability of the duration of life. There is for this two methods of which the result is different; the first, which is that which all Authors have followed, consists in determining this probability by the mean life; that is, by the area of the curve of mortality divided by the number of the living of the same age; see my *Opuscules*, Volume II, page 74 & following.[1] The second, adopted by Mr. de Buffon, is to estimate this probability by the number of years at the end of which the precise half of the living will be dead. I have averted, page 76 of the Work cited, that this is for that which he himself finds such an enormous difference in the first years between the table of mortality of the *Histoire Naturelle* & that of M. *Daniel Bernoulli*, & I know not why this last, after having read that which I have written on this subject, persists to believe (*Mém. Acad. des Sciences de Paris*, 1760, page 28[2]) that the difference comes from a false impression in the table of the *Histoire Naturelle*; although the reason of this enormous difference is evidently that which I have reported. Whatever there is of it, the only difference between these two ways of estimating the probability of the duration of life, would prove that we have not yet at all some sure method for this object; thus I am going to try to show by the following reflections, that one & the other method is subject to some difficulties.

1. And first as to the first method, let two curves of mortality be $AQCD$, $AOCD$, (Fig. 7) of which the areas are equal, but of which the one converges first towards its axis much more promptly than the other; the mean life is the same in the two cases; will we say that the expectation of life is the same? Will we say, that which will be a consequence of it, that two persons, placed in the two cases, could be able to change indifferently from fate the one with the other? It seems to me to the contrary that in the case where the curve of mortality is $AQCD$, the lot is much less favorable; by the reason that there is much less risk of dying in the first years, than when the curve of mortality is $AOCD$.

2. If all men of the same age, & who we suppose to be of number m, lived p years, and who at the end of this time came to perish all at once, their expectation of life, according to the method of which there is question, would be p, & this expectation would be a *certitude*; but they would live in all $2p$ years taking one thing with another, & if there died of them each year an equal number, the expectation of each would be of the same p; now in this

Date: 1768.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH .

[1]This reference is to note (G) to accompany **Memoir 11**, "Sur l'application du Calcul des Probabilités à l'inoculation de la petite Vérole." It is appended to this article.

[2]"Essai d'une nouvelle analyse de la mortalite causee par la petite verole, et des avantages de l'inoculation pour la prevenir."

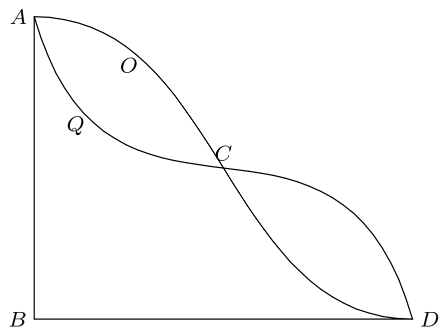


FIGURE 7

last case, the expectation is only a *probability*; can we believe that the two cases are the same? Why therefore estimate the expectation in the two cases by the same number?

3. If concerning the persons of number m, there perished of them in the day or even in the year $\frac{m}{2}$, & if the others lived all to p years, at the end of which they perished all at once, the expectation will be $\frac{p}{2}$, & it will be a simple *probability*. If on the contrary they all lived $\frac{p \text{ years}}{2}$ & if at the end of this time they perished all at once, the expectation would be the same $\frac{p}{2}$, & it would be a *certitude*. Inconvenient similar news in the expression of the expectation; because if the lot is not equal in the two cases, why express it in the last manner?

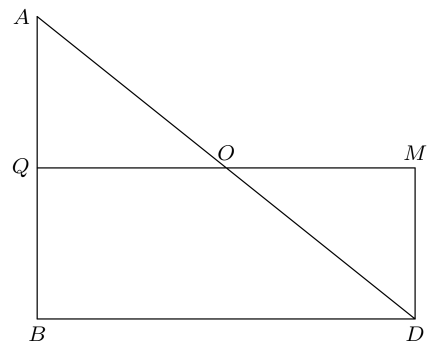
4. We will say perhaps that the disadvantage of having in the last case only a simple *probability* of living $\frac{p}{2}$ years, will be compensated by the *possibility* of living p years; instead that in the second case, we have in truth the *certitude* of living $\frac{p}{2}$ years, but at the same time the *certitude* of not living further. But if the question is to know if this *possibility* of living p years is capable of compensating the fear of dying in the year; in a word, if this is an equal thing, as the result of the calculation gives it, to be assured, for example, of 50 years of life, (neither more nor less) or to have on one side the probability $\frac{1}{2}$ that we will die in the year, or rather in the hour, & on the other the probability $\frac{1}{2}$ that we will live one hundred years.

5. The difficulties are quite similar for the second method. Instead of supposing that the m living persons, die one after the other, so that there remain only $\frac{m}{2}$ at the end of p years, I suppose that they live all p years, & that at the end of this time there die suddenly half of them, that is $\frac{m}{2}$. According to the calculus of the second method, the expectation will be the same in the two cases; but can this be said?

6. In the case of which we just spoke, there is not only *expectation*, there is *certitude* of living p years; in the other there is only *expectation* & not *certitude*; in the first case, beyond the certitude of living p years, we have further the expectation of living to beyond, since we can be of the number of $\frac{m}{2}$ persons who we suppose die only at the end of p years; in the second we have not even the certitude of living p years.

7. On the other hand, we suppose that of m living persons there die suddenly the half of them at the beginning of the p years; by the second method, the probability of the duration of life will be $= 0$, since at the end of a time $= 0$ there is the half of them dying. Now can we say that in this case the *expectation* is $= 0$? Indeed, we could suppose (Fig. 8) that after the half AQ of living persons AB is dead suddenly at the beginning of the time $BD = p$

years, the entire half remaining QB live one hundred years, & die only at the end of this time. Now in this case could we not say: there are odds one against one that I will live one hundred years or that I will die just now; therefore my *expectation* is fifty years.



8. Of these two methods to estimate the probability of life, the first is absolutely analogous to the calculation by which we determine the expectation of the Players in the games of chance; thus it is followed by a much greater number of Authors than the second, which nonetheless can have also its partisans. If we consider the expectation of living following the idea of the first method, it seems to me that the difficulty is of knowing how we must estimate the life by regarding it as *wealth*, as a sum taken in a *game*.

9. If we suppose a Lottery where after the drawing the half of the living die suddenly, & the other live 100 years, 1000 years, &c. the expectation will be 50, 500 years, &c. Who is the man who would wish to put into this Lottery, & who believed, by putting into it, to render his good lot worse, although by remaining in the ordinary state, his *expectation* of living, to whatever age that this be, is less than 50 years?

10. Now why in the first case is the *lot* really more disadvantageous than in the second, which is the ordinary state, although the calculus gives in the first case the greater *expectation*? It is that in the second case the risk of dying is shared over a long space of time, & that it is slight enough in each small part of this long time; instead that in the second case, this risk is found suddenly $\frac{1}{2}$ in a very short time; a consideration which must enter into the calculation, as all men likewise will make enter implicitly, & which nevertheless all the calculators have neglected.

11. It seems to me therefore that in every calculation on the estimation of life, we have not had enough regard to one thing, to the time which must be elapsed between the moment where we live, & to that where we can die; because, as I have already observed besides, the risk of dying is so much less, all things equal besides, when we must live a longer time before succumbing to this risk; a consideration which is here very essential, & which puts especially a great weight in the balance, when there is question of loss of life immediately or in a few days. See on this subject the *Réflexions sur l'inoculation*, Volume II of my *Opuscules Mathématiques*, & Volume V of my *Mélanges de Philosophie*.

(G) 1. Before expanding this difficulty, it will not be useless to propose another on it, which is general for the estimation of mortality. It falls on the manner of preparing the degrees of probability of life. If we can hold ourselves on that of the ordinary rules of probabilities, & if we regard life as a kind of Lottery or game of chance, we will find that the *expectation* of each Player or man, is equal to the sum of the living persons at the end of each year AR (fig. 1.) divided by the number AK of living persons at the beginning A of the time AQ; that which gives the entire area $AKEQ$ divided by AK: that is, that the *expectation* of each man is equal to the times which all these men must live taken together, these times being divided by the number of men; as in a Lottery where each player has taken a ticket, the *expectation* of each player is equal to the sum of the lots divided by the number of tickets. It seems therefore, following this first manner so natural to consider the thing, that the times that each man can expect to live, must be counted equal to that which we call commonly, *his mean life*.

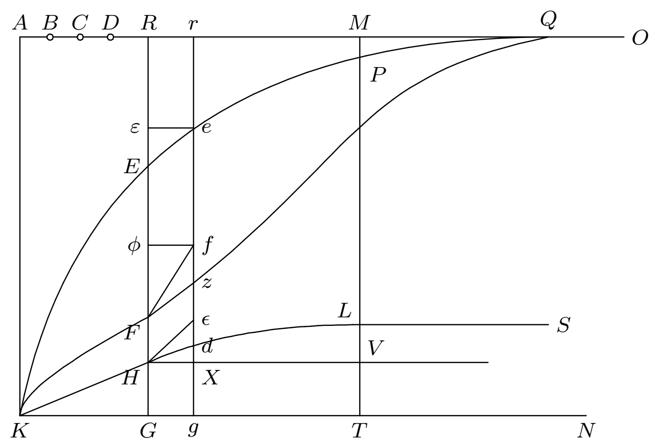


FIGURE 1

2. However there is another way also entirely plausible to consider the question, which gives another result. This is to seek the times AR, at the end of which there will die the half of the living AK; & to regard these times as the one which we can expect to live: since we can wager evenly or one against one, who will be yet living at the end of these times. This time AR is different from the one which gives the *mean life*; excepting in a single case which has no place in nature: this is the case where KEQ would be a straight line, that is, where there would die each year an equal number of persons. Now which must be preferred of these two ways to estimate the duration of life? Both would appear equally plausible, although they give some very different results. For example, the duration of life of newborn infants, is estimated, according to the first method, at 26 years nearly by the calculations of Mr. Halley; & the duration of life of these infants, estimated according to the second method, is around 8 years. (*See the Table gathered at the end of the second Volume of the Histoire Naturelle of Messrs. de Buffon & d'Aubenton*). There comes from this that there die a prodigious quantity of infants in the first year of life.

3. By supposing this first difficulty resolved, that which we have touched in our Memoir[3], will subsist further in all its force. We suppose that a be the *expectation of life*, either the duration of life, estimated from the one or the other of the two preceding manners; & that $a+c$ be the *expectation of life* for the inoculated. It is clear 1° that the one which makes

[3]*Translator's note*: The question as to whether the immediate risk of death incurred by inoculation of the small pox outweighs the long term risk of death without inoculation.

himself inoculated, acquires the expectation of life after the time a, a number of years $= c$; 2°. that he risks $\frac{1}{300}$, or, if we wish, in general $\frac{1}{n}$ to sacrifice in a month, in 15 days, &, so to speak, suddenly (because the one returns nearly to the same for a time so short) the entire time a that he can expect to live. We could therefore regard $-\frac{a}{n}$ as the risk, & c as the expectation, if all things were equal besides. But it is necessary to remark 1°. that the risk $-\frac{a}{n}$ is incurred in the month, & for thus to say in the day; instead that the expectation of life a number c years, is rejected at the end of time a. And even when we would not regard the expectation c as diminished by the time a at the end of which it is placed, we can scarcely conceal ourselves that the risk $-\frac{a}{n}$ is not increased by the little time during which it is incurred, especially when the question is of life, that is, of the most precious of all goods. Now by what reason is the risk $-\frac{a}{n}$ increased by this briefest of times? This is on what we can only make some hypotheses. 2° If the time a, at the end of which the years of expectation c take place, attains up to an advanced age, as of 60 years & more, it is evident, that during the years c, we will be subject to the infirmities of agedness; & that thus the expectation c must be diminished in this regard: since the times which we permit, is properly a time to subtract on the true duration of life, on the life properly so-called. Now according to what law must this quantity c be diminished? It is again on what we can only make some hypotheses, always vague & little satisfying.