SUITE DU MÉMOIRE SUR LE CALCUL DES PROBABILITS*

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THIRD PART
On the evaluation of possible Claims

The destruction of the feudal Government has permitted to subsist in Europe a great number of possible claims, but one can reduce them to two principle classes; the ones are paid when property just changes by sale, the others are paid by the transfers at succession, either direct or collateral, or collateral alone.

One has regarded the first kind of claim as an obstacle to the sale of properties, & consequently to the improvement of the estates: the rights of the second kind have appeared an importune & often ruinous hindrance. One has claimed also that the owners of these rights would find advantage to exchange them against an annual revenue, but a person, who I know, is not himself occupied in the ways to evaluate these claims, this work would have however some utility; indeed, it would give to the private persons who would wish either to sell or to purchase these claims, a fixed base according to which they could negotiate; & in the case where a Government would wish to order the refund, one would draw from it the means to understand the advantages of this operation, & those to execute it with justice. Finally, the possible claims are a property, a true revenue which can be subject to a tax; they can be regarded also as diminishing the true yield of the estates which is burdened by it, & their evaluation, under this point of view, can yet be useful.

We will limit ourselves uniquely here to that which regards the calculus, & we will give only some general formulas which can be applied to all kinds of Claims, to all the principles of Jurisprudence or of Administration according to which we can make evaluation of it.

We will begin by examining the case in which the transfer, or rather generally the event which produces the claim, arrives necessarily after a certain space of time, as the one where one owes a claim for each direct succession or not; next we will consider

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 12, 2009 the one where this event is not necessary, as when the claim is due for a sale or for a single kind of succession; we will examine next these evaluations relative to the one who possesses the thing subject to the claim: finally we will suppose that a very same wealth is subject to two different claims which it is necessary to evaluate.

I.

First principle. We will suppose first that the order according to which the last transfers are themselves succeeded, will be indefinitely continued.

The motive which has made us adopt this principle, is the great probability that we have fewer great changes, fewer great revolutions to expect for the future, than there have been in the past: the progress of knowledge of each kind & in all the parts of Europe, the spirit of moderation & of peace which reigns there, the kind of contempt where the Machiavellian begins to fall, seems to assure us that the wars & the revolutions will become in the future less frequent; thus the principle which we adopt, at the same time as it renders the calculus & the observations more easy, has greater advantage to be more exact.

Second principle. One will regard the changes as equally probable, whatever be the value, the nature, the situation of the properties, the tax & the form of the claim to which they are subject. It is possible that Observation makes to discover great differences between the diverse kinds of properties; but then it would be necessary to classify the claims or the properties, & to make in part the calculation for each class; thus this second principle must be admitted generally.

We will resolve first the problem by supposing that the claim is due, & that the event or the transfer has actually taken place; & next we will give the means to apply the calculus to the case where one would wish to make the evaluation for a period placed between two transfers; we will give for this problem three methods based on some different ways to envision the question, of which each can in certain circumstances merit to be preferred.

First method.

Let a', a'', $\ldots a'''^n$ be the numbers of the elapsed years between two observed transfers;[1] b', $\ldots b'''^n$ the numbers of transfers corresponding to these intervals of a', a'', $\ldots a'''^n$ years; 1 the value of the claim for any property whatever at the moment of the transfer, $\frac{1}{m}$ the annual interest of the claim 1; & let one demand the total value of the claim, as much for the actual transfer as for all the future transfers, this value being carried back to the present time. One knows that the claim 1 which would be due only at the end of z years, would be then expressed by $\left(\frac{m}{m+1}\right)^z$, or, for brevity, by c^z.

Let there be therefore a number p of successive transfers, of which p' are arrived at the end of a' years, p'' at the end of a'' years $\ldots p'''^n$ at the end of a'''^n years. It is clear that in whatever order that these transfers are themselves succeeded, the last will

[1] *Translator's note*: Condorcet uses an awkward prime notation whereas today we would use subscripts. Thus he writes a'''^n to indicate a_n. I have retained his notation throughout.

arrive at the end of $p'a' + p''a'' + p'''a''' \cdots + p'''^{n}a'''^{n}$ years; so that the sum due for this transfer will be always

$$c^{a'p' + a''p'' + a'''p''' \cdots + a'''^{n}p'''^{n}}$$

If next we call x' the probability of the transfer after a' years, x'' the probability of the transfer after a'' years, x'''^{n-1} the probability of the transfer after a'''^{n-1} years, finally $1 - x' - x'' \cdots - x'''^{n-1}$ the probability of the transfer after a'''^{n} years, the probability of this p^{th} transfer that we just considered, will be expressed, by

$$\frac{1.2.3\ldots p}{1.2\ldots p' \times 1.2\ldots p'' \times 1.2\ldots p''' \times \ldots 1.2\ldots p'''^{n}} \times x'^{p'} x''^{p''} x'''^{p'''} \cdots (1 - x' - x'' \cdots x'''^{n-1})^{p'''^{n}}$$

so that the value of all the p^{th} transfers, each multiplied by their respective probability, will be

$$\Big[c^{a'}x' + c^{a''}x'' + c^{a'''}x''' \cdots + c^{a'''^{n}}\,(1 - x' - x'' - x''' \cdots - x'''^{n-1})\Big]^{p}$$

this which represents the mean value of the claim of this transfer.

But here the x are not some given and constant quantities. One knows only that the event of which the probability is expressed by x', is arrived b' times; that the one of which the probability is expressed by x'', is arrived b'' times, & thus in sequence; the mean value of the claim for the p^{th} transfer, will be therefore expressed by

$$\frac{\int \left\{ x'^{b'} x''^{b''} \cdots (1 - x' \cdots - x'''^{n-1})^{b'''^{n}} \left[c^{a'}x' + c^{a''}x'' \cdots + c^{a'''^{n}}(1 - x' \cdots - x'''^{n-1})\right]^{p} \partial x' \partial x'' \cdots \partial x'''^{n-1} \right\}^{n-1}}{\int \{x'^{b'} x''^{b''} \cdots (1 - x' - x'' \cdots - x'''^{n-1})^{b^{n}} \partial x' \partial x'' \cdots \partial x'''^{n-1}\}^{n-1}}$$

the integration being repeated an $n-1$ number of times, & the integrals taken from $x'''^{n-1} = 0$ to $x'''^{n-1} = 1 - x \cdots - x'''^{n-2}$; from $x'''^{n-2} = 0$ to $x'''^{n-2} = 1 - x' \cdots - x'''^{n-3}$; ... from $x'' = 0$ to $x'' = 1 - x'$; from $x' = 0$ to $x' = 1$.

The denominator of this function, is the same for all the values of p, & the numerator forms a geometric series. The value of the claim for all the future transfers, by counting from the actual transfer, will be therefore expressed by the formula

$$\frac{\int \left\{ \frac{x'^{b'} x''^{b''} x'''^{b'''} \cdots (1 - x' \cdots - x'''^{n-1})^{b'''^{n}}}{1 - c^{a'}x' - c^{a''}x'' - c^{a'''^{n}}(1 - x' \cdots - x'''^{n-1})} \times \partial x' \partial x'' \cdots \partial x'''^{n-1} \right\}^{n-1}}{\int \{x'^{b'} x''^{b''} x'''^{b'''} \cdots (1 - x' - x'' \cdots - x'''^{n-1})^{b'''^{n}} \partial x' \partial x'' \cdots \partial x'''^{n-1}\}^{n-1}},$$

the integrals being taken as above.

If one supposes that the claim was not due at the actual moment, & if it has not been paid at all after α years; let a', a'', a''', &c. be supposed ranked according to their order of greatness, & $a' < a'' < a''' \cdots$, & let $\alpha > a'''^{m}$ & $< a'''^{m+1}$, the sought value will be expressed by the formula

$$\frac{\left\{ \begin{array}{l} \int \Big\{ x'^{b'} x''^{b''} \cdots (1 - x' \cdots - x'''^{n-1})^{b'''^{n}} \times \\ \frac{c^{a'''^{m+1} - \alpha} x'''^{m+1} + c^{a'''^{m+2} - \alpha} x'''^{m+2} \cdots + c^{a'''^{n} - \alpha}(1 - x' \cdots - x'''^{n-1})}{1 - c^{a'}x' - c^{a''}x'' \cdots - c^{a'''^{n}}(1 - x' \cdots - x'''^{n-1})} \times \\ \qquad \partial x' \partial x'' \cdots \partial x'''^{n-1} \Big\}^{n-1} \end{array} \right\}}{\int \{x'^{b'} x''^{b''} \cdots (1 - x' - x'' \cdots - x'''^{n-1})^{b'''^{n}} \left[x'''^{m+1} + x'''^{m+2} \cdots + (1 - x' \cdots - x'''^{n-1})\right] \partial x' \partial x'' \cdots \partial x'''^{n-1}\}^{n-1}}$$

which differs from the preceding, only because one has multiplied the numerator under the sign by

$$c^{a'''^{m+1}-\alpha}x'''^{m+1}+c^{a'''^{m+2}-\alpha}x'''^{m+2}\cdots+c^{a'''^{n}-\alpha}(1-x'\cdots-x'''^{n-1}),$$

& the denominator, by the function

$$x'''^{m+1}+x'''^{m+2}\cdots+(1-x'\cdots-x'''^{n-1}),$$

or by the function $1-x'\cdots-x'''^{m}$ which is equal to it.

If the b are very great numbers, this which is necessary besides, if one wishes to have some assurance that the mean value thus determined, differs little from the true value; one can substitute in the first formula

$$\frac{1}{1-\frac{c^{a'}b'+c^{a''}b''\cdots+c^{a'''^{n}}b'''^{n}}{b+b'+b''\cdots+b'''^{n}}}$$

or more exactly

$$\frac{1}{1-\frac{c^{a'}(b'+1)+c^{a''}(b''+1)\cdots+c^{a'''^{n}}(b'''^{n}+1)}{b+b'+b''\cdots+b'''^{n}+n}}$$

& in the second this same formula, multiplied by

$$\frac{c^{a'''^{m+1}-\alpha}b'''^{m+1}+c^{a'''^{m+2}-\alpha}b'''^{m+2}\cdots+c^{a'''^{n}}b'''^{n}}{b^{m+1}+b^{m+2}\cdots+b'''^{n}},$$

or by

$$\frac{c^{a'''^{m+1}-\alpha}(b'''^{m+1}+1)+c^{a'''^{m+2}-\alpha}(b'''^{m+2}+1)\cdots+c^{a'''^{n}}(b'''^{n}+1)}{b'''^{m+1}+b'''^{m+2}\cdots+b'''\,^{n}+n-m}.$$

One has regarded as possibles in the preceding method only the transfers which arrive at the end of the same intervals of time a', a'', $\ldots a'''^{n}$ years, for which these transfers have been observed. This assumption can appear rigorous only in the case where these transfers have taken place for nearly all the possible intervals from a' to a'''^{n} years. We will propose therefore another method, in which one will suppose them possible after one year, 2, 3, &c. years.

Second method.

We will maintain here the same denominations as above. This put, let x be the probability of the transfer at the end of one year; $(1-x)x$ will be this probability at the end of two years, $(1-x)^2x$ at the end of three years, & so forth; so that $cx+(1-x)c^2x+(1-x)^2c^3x+(1-x)^3c^4x+$&c. will express the value of the claim for the first transfer which must take place; & summing the series, this value will be expressed by $\frac{cx}{1-c+cx}$; for the second transfer it will be $\frac{c^2x^2}{(1-c+cx)^2}$; for the third, $\frac{c^3x^3}{(1-c+cx)^3}$, & so forth. Adding therefore to these terms 1, the value of the transfer that one supposes to have taken place, & to be due at the instant where one seeks to evaluate the claim, one will have, by taking the sum of the series,

$$1+\frac{cx}{1-c+cx}+\frac{c^2x^2}{(1-c+cx)^2}+\text{\&c.}=\frac{1-c+cx}{1-c},$$

& the total value of the claim will be expressed by the formula

$$\frac{\int\left\{(1-x)^{(a'-1)b'+(a''-1)b''\cdots+(a'''n-1)b'''n}x^{b'+b''\cdots+b'''n}\left(\frac{1-c+cx}{1-c}\right)\partial x\right\}}{\int\left\{(1-x)^{(a'-1)b'+(a''-1)b''\cdots+(a'''n-1)b'''n}x^{b'+b''\cdots+b'''n}\partial x\right\}},$$

the integral being taken from $x=0$ to $x=1$.

And because $\frac{1-c+cx}{1-c}=1+\frac{cx}{1-c}$, this formula will be expressed by

$$1+\frac{c}{1-c}\cdot\frac{b'+b''\cdots+b'''n+1}{a'b'+a''b''\cdots a'''nb'''n+2}.$$

If one does not suppose the claim due, & if there are α years elapsed from the last transfer, instead of the preceding formula, one will have for the expression of the value,

$$\frac{\int\left\{(1-x)^{(a'-1)b'+(a''-1)b''\cdots+(a'''n-1)b'''n}x^{b'+b''\cdots+b'''n}\frac{cx}{1-c}\partial x\right\}}{\int\left\{(1-x)^{(a'-1)b'+(a''-1)b''\cdots+(a'''n-1)b'''n+\alpha}x^{b'+b''\cdots+b'''n}\partial x\right\}}$$
$$=\frac{c}{1-c}\times\frac{b'+b''\cdots+b'''n+1}{a'b'+a''b''\cdots a'''nb'''n+\alpha+2}$$

In this method, one supposes that all the transfers observed are equally probable, & that they have always been in all the course of the duration; but one can also admit the contrary hypothesis, that is to say, to suppose the probability different for the different intervals observed in the transfers, that which leads us to the third method.

Third method.

Maintaining always the same denominations, we will call z', z'', $\ldots$, $1-z'-z''\cdots z'''n-1$, or $z'''n$ the probabilities that the event for the succession of which one seeks the value of the claim, will be in the list of events of which the transfer is arrived at the end of a', a'', $\ldots a'''n$ years, & x', x'', x''', $\ldots x'''n$ the unequal probabilities for the corresponding transfers at each interval. In this case, one can suppose, or that in the sequence of events the one that one considers will belong always to the same z', or can appear successively in each; in the first hypothesis,

The expression of the mean value of the claim will be

$$z'\cdot\frac{1-c+cx'}{1-c}+z''\cdot\frac{1-c+x''}{1-c}\cdots+z'''n\cdot\frac{1-c+cx'''n}{1-c},$$

and consequently the formula which represents the claim will be

$$\frac{\left\{\begin{array}{r}\int\left\{\left[z'^{b'}z''^{b''}\cdots(1-z'-z''\cdots-z'''^{n-1})^{b'''n}\times\right.\right.\\ z'\left(1+\frac{c}{1-c}\times\frac{b'+1}{a'b'+2}\right)+z''\left(1+\frac{c}{1-c}\times\frac{b''+1}{a''b''+2}\right)\cdots\\ \left.\cdots+(1-z'-z''\cdots-z^{n-1})\left(1+\frac{c}{1-c}\times\frac{b'''n+1}{a'''^{n}b'''^{n}+n}\right)\right]\times\\ \left.\partial z'\partial z''\cdots\partial z'''^{n-1}\right\}^{n-1}\end{array}\right\}}{\int z'^{b'}z''^{b''}\cdots(1-z'-z''\cdots-z'''^{n-1})^{b'''n}\partial z'\partial z''\cdots\partial z'''^{n-1}}.$$

If one supposes now that the same event can appear successively in all these classes, then the mean value of the claim will be

$$\frac{1}{1-\frac{cz'x'}{1-c+cx'}-\frac{cz''x''}{1-c+cx''}\cdots-\frac{cz'''^{n}x'''^{n}}{1-c+cx'''^{n}}},$$

one will form the mean value of this formula for all the values of x, taken for each x, from $x=0$ to $x=1$, & let Z be this value, the formula which will represent, will be expressed by

$$\frac{\int\left\{z'^{b'}z''^{b''}\cdots(1-z'-z''\cdots-z'''^{n-1})^{b'''n}Z\partial z'\partial z''\cdots\partial z'''^{n-1}\right\}^{n-1}}{\int\left\{z'^{b'}z''^{b''}\cdots(1-z'-z''\cdots-z'''^{n-1})^{b'''n}\partial z'\partial z''\cdots\partial z'''^{n-1}\right\}^{n-1}},$$

the integrals being taken from $z'''^{n-1}=0$, to $z^{n-1}=1-z'\cdots-z'''^{n-2}$, ... from $z''=0$ to $z''=1-z'$, & $z'=0$ to $z'=1$.

Under these same hypotheses, if one seeks the value for the claim in the case where the transfer has not taken place after α years, it will suffice to put into the preceding formula $a'b'+\alpha$, $a''b''+\alpha$, &c. instead of $a'b'$, $a''b''$, &c. & to subtract unity.

We will say nothing more on these formulas, if this is only they are integrated by the known methods, & if besides one would have some approximate values, either by the method given by Mr. Euler, or by those which Mr. de la Place has exposed in this same volume.[2]

But we will add some observations on the hypotheses which we have followed: first we have supposed all the periods annual, this assumption is not rigorously exact, but it would become it, if instead of supposing that the a represent the years, one takes them for the halves or the quarters of a year; then the errors which could result from this manner of treating the question, would be very small, & perhaps would approach more to the truth, which if one sought a greater exactitude because in the arrangements of this kind, the rigorous assumption of the compound interests always increasing, extends itself too much from ordinary usage. We suppose therefore one calculates from three months to three months: let c be the value for the year, it will be necessary that c' being the value for the quarter, $c'^4=c$; in this way one will suppose the claims always payable from three months to three months; & as the delay which it is often necessary to accord, & sometimes the acceleration of the prescribed term, without that those who must calculate the small loss of interest, produce a sort of variation in the real term

[2] *Translator's note*: "Sur les approximations des Formules qui sont fonctions de très-grands nombres."

of the payments to make, this hypothesis appears to us sufficiently exact. One would suppose equally that if c is the value of a sum 1 a year before its deadline, $c' = c^{\frac{1}{4}}$ is the value of the same sum three months before the deadline: thus supposes that in the fraction of the year which can take place, the one who would have received the claim in advance, would have placed it from three months to three months in this new interest, which represents the annual interest, & which is a little less than that interest; one would give him therefore a little more that he would have had in the case of the simple annual interest, but this excess is a very small thing, & would be compensated by the loss of the time which one must suppose also between a reimbursement & a new placement.

If one has made a very great number of observations, it is very probable that one will have made them from three months to three months, that thus the a', a'', $\ldots a'''^{n}$ will represent all the possible intervals between the transfers, excepting some extraordinary cases where the transfers would be either very extended or very near; then the first method can be employed: the second supposes more, that all the observed events were equally probable, a supposition which, if one has made the observations on some events of the same nature, is very admissible; & it supposes still that each year the transfer is equally probable: this second supposition appears equally in the third method which supposes besides the probability only the same for the similar events &, if one admits the second formula, for one transfer only. This last method appears therefore more rigorous, & it must be preferred for all the cases where one besides would not have reasons to believe the probability the same for all the events; & one would prefer the first or the second hypothesis, according as one would have cause to suppose the difference between the probabilities of the events, either constant for the same class, or variable in general.

Finally these last two methods would be acceptable in the case where the question is of purely accidental transfers, as some sales, or even as certain claims due to the marriage of the lord, to the birth of his eldest son, to some purely collateral successions, even to the direct successions, by supposing that it is not due in case of sale, &c. but not in the case, for example, of claims due to the death of such individuals, landlords of inalienable wealth, since the probability of this claim increases then necessarily, in measure as one extends from a certain period; instead that, for example, if one has not paid at all before the hundredth year the claim due for a collateral succession or a direct succession, because of alienation, there is no reason to believe more probable, in general, that the event arrives in this one hundred first year than in any other.

We will observe that there are some cases where one must, whatever hypothesis that one takes, follow a different method. We suppose, for example, that the question is of a claim on the successions, & that there is an actual possessor to which the alienation is forbidden; it is clear that it will be necessary to take the total value of the claim, & to multiply it by the sum of the probabilities that he will die at the end of $a'_{\prime}$, $a''_{\prime}$, $a'''_{\prime}$, &c. intervals of time, each multiplied by $c^{a'_{\prime}}$, $c^{a''_{\prime}}$, $c^{a'''}$, &c.

II.

We will consider now the case where the events which produce the claim, can be supposed to be arrived not at all in a certain number of circumstances. The last two methods have no difficulty; indeed, it will suffice in the second method, if one has a sequence of events of which $b'_{\prime}$, $b''_{\prime}$, $b'''_{\prime}$, &c. are the number, & $a'_{\prime}$, $a''_{\prime}$, $a'''_{\prime}$, &c. the number of years that they have passed without arriving, to add $a'_{\prime}b'_{\prime}+a''_{\prime}b''_{\prime}+a'''_{\prime}b'''_{\prime}+$&c. to the denominator of the formula, & as for the third, to suppose moreover a sequence of $z'_{\prime}$, $z''_{\prime}$, $z'''_{\prime}$, &c. corresponding to these sequences of $a_{\prime}$ & $b_{\prime}$. In the first method, a difficulty appears moreover; indeed it is necessary to distinguish two cases, 1.° the one where these events are not yet arrived, but where the interval between two events is less than a'''^{n} which one regards here as the last term; in this case let $b'_{\prime}$ be the number of those which correspond to a number a'''^{m} of years, the probability that they would arrive after a'''^{m+1}, a'''^{m+2}, &c. years will be expressed by

$$\frac{a'''^{m+1}}{1-a'\cdots a'''^{m}}, \quad \frac{a'''^{m+2}}{1-a'\cdots a'''^{m}}, \quad \text{\&c.}$$

& thus it will suffice to multiply the numerator & the denominator under the sign, by

$$(x'''^{m+1}+x'''^{m+2}+x'''^{m+3}\cdots+x'''^{n})^{b'_{\prime}}.$$

We suppose next that one has a certain number of cases in which the event, after a certain number of years greater than a'''^{n} has taken place, one can, under this same hypothesis, regard them only as some particular events which produce no claim; this put, let $x'_{\prime}$, $x''_{\prime}$, $x'''_{\prime}$, $\cdots x'''^{n}_{\prime}$ be their probabilities, $b'_{\prime}$, $b''_{\prime}$, $b'''_{\prime}$, $\cdots b'''^{n}_{\prime}$ their number, $a'_{\prime}$, $a''_{\prime}$, $a'''_{\prime}$, $\cdots a'''^{n}_{\prime}$ the one of the years which correspond to it; it is easy to see 1.° that it will be necessary in the denominator of the formula, & in the factor of the numerator which multiplies the expression of the value of the claim to have regard to these x precisely as to those of the other series; 2.° that in order to determine the expression of the value of the claim, it will be necessary in the place of $\frac{1}{1-c^{a'}x'\cdots-c^{a'''^{n}}x'''^{n}}$, to take

$$\frac{1-c^{a'_{\prime}}x'_{\prime}-c^{a''_{\prime}}x''_{\prime}\cdots-c^{a'''^{n}_{\prime}}x'''^{n}_{\prime}}{1-c^{a'}x'-c^{a''}x''\cdots-c^{a'''^{n}}x'''^{n}-c^{a'_{\prime}}x'_{\prime}-c^{a''_{\prime}}x''_{\prime}\cdots-c^{a'''^{n}_{\prime}}x'''^{n}_{\prime}},$$

by supposing the last x equal to unity less all the other x.

III.

The possible claim is for the one for which a kind of fancier property is due which has a value, & the reimbursement of this claim is another property which one changes against the first. If therefore their values are equals, he will sustain neither loss, nor gain by the exchange. It is clear at the same time that this claim is a debt for the one who is subject to pay it; but what is the nature of this debt? And what is charged to it?

We suppose, for example, a claim due solely for the successions. It is clear first that these are only the inheritances of actual landholders, & thus in sequence, from generation to generation; a landowner who would reimburse this claim would make therefore precisely the same thing as if he placed an equivalent sum of which the principle & interests always increasing would be destined to his heirs without that he ever enjoyed it; this is therefore not his debt which he would pay, it is that of his children, of his heirs; the reimbursements of these claims must therefore be made only voluntarily by him; so that if one judges these claims injurious, it is at the expense of the Public which they must be made.

One must observe however that the claim of this kind diminishes the value of the property, & in this regard the suppression of the claim would make him win, not on the revenue, but on the land of the estate.

If the claim is due on a sale, it becomes then dependent on the will of the one who pays it, there results from it then necessarily a rebate more or less strong; thus this is according to the mean value of the claim thus reduced, that must be made the evaluation.

In this same case, the landowner of the wealth who owes this claim, has interest in that which he is abolished; the suppression of the claim will increase the value of the wealth without increasing the revenue, as for the claim due to the successions; but in the one & in the other case this increase of value is not equal to that of the claim, & it would be much more feeble in the first.

There would be therefore always a difference between the value of the claim for the one who collects it, & the value of the same claim for the one who pays it. The voluntary reimbursement would be therefore rare, & would take place only under some particular circumstances. For the same reason one could not with justice oblige; thus in the case where one would judge these rights injurious it would be necessary either to reimburse them at the expense of the public treasury, or to facilitate the voluntary reimbursements by paying a part of the value.

IV.

We suppose now two claims S & V, for one of which the value of the claim is 1, & D for the second, c expressing the value of the claim 1, if it is due only at the end of one year; so that if it is due at the end of z years, this value is c^z, & Dc' expressing the value of the second claim, if it is due only at the end of one year; so that if it is due only at the end of z years, this value is Dc'^z.

If we employ the first method, & if we suppose that the intervals elapsed between the payment of two claims, the number of the observations for each interval, & the probability of each they are represented by

$$\left.\begin{array}{l} a',\, a'',\, \cdots a'''^n \\ b',\, b'',\, \ldots b'''^n \\ x',\, x'',\, \ldots x'''^n \end{array}\right\}\text{for the case where } S \text{ succeeds to } S,$$

$$\left.\begin{array}{l} a'_{\prime},\, a''_{\prime},\, \cdots a'''^{n'}_{\prime} \\ b'_{\prime},\, b''_{\prime},\, \ldots b'''^{n'}_{\prime} \\ x'_{\prime},\, x''_{\prime},\, \ldots x'''^{n'}_{\prime} \end{array}\right\}\text{for the case where } V \text{ succeeds to } S,$$

$$\left.\begin{array}{l} a'_{\prime\prime},\, a''_{\prime\prime},\, \cdots a'''^{n''}_{\prime\prime} \\ b'_{\prime\prime},\, b''_{\prime\prime},\, \ldots b'''^{n''}_{\prime\prime} \\ x'_{\prime\prime},\, x''_{\prime\prime},\, \ldots x'''^{n''}_{\prime\prime} \end{array}\right\}\text{for the case where } S \text{ succeeds to } V,$$

$$\left.\begin{array}{l} a'_{\prime\prime\prime},\ a''_{\prime\prime\prime},\ \cdots a'''^{n'''}_{\prime\prime\prime} \\ b'_{\prime\prime\prime},\ b''_{\prime\prime\prime},\ \ldots b'''^{n'''}_{\prime\prime\prime} \\ x'_{\prime\prime\prime},\ x''_{\prime\prime\prime},\ \ldots x'''^{n'''}_{\prime\prime\prime} \end{array}\right\} \text{for the case where } V \text{ succeeds to } V.$$

We seek first the value of the claim for a p^{th} transfer: for this, we will make, for brevity,

$$x' + x'' \cdots + x'''^{n} = E, \quad x'_{\prime} + x''_{\prime} \cdots + x'''^{n}_{\prime} = F,$$
$$x'_{\prime\prime} + x''_{\prime\prime} \cdots + x'''^{n}_{\prime\prime} = G, \quad x'_{\prime\prime\prime} + x''_{\prime\prime\prime} \cdots + x'''^{n}_{\prime\prime\prime} = H;$$

& naming in the term which expresses the probability of the p^{th} transfer, P the part which is terminated by the event which produces the claim S, & Q the part which is terminated by the event which produces the claim V, we will have

$$PF + QG = P', \quad \& \quad PF + QH = Q';$$

P'& Q' being that which P & Q becomes when p becomes $p + 1$, we will draw from it the equation

$$P'' - (E + H)P' = (GF - HE)P;$$

therefore making $P = Ar^p + Bs^p$, we will have

$$r = \frac{E + H}{2} + \frac{1}{2}\sqrt{(E - H)^2 + 4GF},$$

&

$$S = \frac{E + H}{2} - \frac{1}{2}\sqrt{(E - H)^2 + 4GF}, \quad r = z + \sqrt{u}, \quad s = z - \sqrt{u}.$$

One will have equally $Q = A'r^p + B's^p$; but $Q = \frac{P' - PE}{G}$; consequently one will have $A = \frac{Ar - AE}{G}$, $B' = \frac{Bs - BE}{G}$.

We suppose now that one part of the moment where the event which corresponds to S has taken place, one will have for when

$$p = 0, \quad P = A + B = 1, \quad \& \, Q = A' + B' = 0,$$

& consequently $A = \frac{F - s}{r - s}$, $B = \frac{r - F}{r - s}$. The sum of all the P will be therefore, by setting for r & s their values,

$$\frac{1 + E - z}{(1 - z)^2 - u}, \quad \text{or} \quad \frac{1 - H}{1 - E - H + HE - GF};$$

likewise one will have

$$Q = \frac{F}{1 - E - H + HE - GF};$$

this put, if we call Π and Φ the preceding values of P & Q, by putting into P for x, xc^a, each a being the one of the a which corresponds to each x, & into Q for x, xc'^a, a being always the one of the a which corresponds to each x, one will have the value of the double claim expressed by the formula

$$\frac{\left\{\begin{array}{r}\displaystyle\int\Big\{\Big(x'^{b'}\cdots x'''^{nb'''n}\cdot x_{\prime}'^{b_{\prime}'}\cdots x'''^{b_{\prime}'''n'}\cdot x_{\prime\prime}'^{b_{\prime\prime}'}\cdots x_{\prime\prime}'''^{b_{\prime\prime}^{n''}}\cdot x_{\prime\prime\prime}'^{b_{\prime\prime\prime}'}\cdots x_{\prime\prime\prime}'^{b_{\prime\prime\prime}'''n'''}\Big)\times\\ (\Pi+D\Phi)\,\partial x'\,\partial x''\text{ \&c.}\Big\}^{\,n+n'+n''+n'''-2}\end{array}\right\}}{\left\{\begin{array}{r}\displaystyle\int\Big\{\Big(x'^{b'}\cdots x^{nb^{n}}\cdot x_{\prime}'^{b_{\prime}'}\cdots x_{\prime}'''^{nb'''n'}\cdot x_{\prime\prime}'^{b_{\prime\prime}'}\cdots x_{\prime\prime}'^{b_{\prime\prime}'''n''}\cdot x_{\prime\prime\prime}'^{b_{\prime\prime\prime}'}\cdots x_{\prime\prime\prime}'^{b_{\prime\prime\prime}'''n'''}\Big)\times\\ \partial x'\,\partial x''\text{ \&c.}\Big\}^{\,n+n'+n''+n'''-2}\end{array}\right\}}$$

in which formula, one will substitute to

$$x_{\prime}'''^{n'},\ 1-(x'+x''\cdots+x^{n}+x_{\prime}'+x_{\prime}''\cdots+x_{\prime}'''^{n'-1}),$$

& to

$$x_{\prime\prime\prime}'''^{n'''},\ 1-(x_{\prime\prime}'+x_{\prime\prime}''\cdots+x_{\prime\prime}'''^{n''}+x_{\prime\prime\prime}'+x_{\prime\prime\prime}''\cdots+x_{\prime\prime\prime}'''^{n'-1}),$$

& we will take the integrals separately for the x contained in $E+F$, & in $G+H$, in a manner that they are extended from zero to unity for each class, as in the first article.

The case where any of the claims would be due, will be solved in the same manner as the analogous case of the first article.

If one employs the second method, one will observe that one can apply equally the preceding formulas, & that there will be another change to make only to substitute into these formulas, instead of E, F, G & H, that which becomes in these methods the probabilities that S will succeed to S, or V to S, or S to V, or V to V. One will take therefore x the probability that S will succeed to S in the year, x' that V will succeed to S, $1-x-x'$ that neither one nor the other will succeed in the year; likewise $x_{\prime}$ will express the probability that S will succeed to V, & $x_{\prime}'$ that V will succeed to V. We will have therefore

$$E=\frac{x}{1-(1-x-x')};\qquad F=\frac{x'}{1-(1-x-x')};$$
$$G=\frac{x_{\prime}}{1-(1-x_{\prime}-x_{\prime}')}\qquad H=\frac{x_{\prime}'}{1-(1-x_{\prime}-x_{\prime}')}$$

& it will be necessary to from Π & Φ, to multiply in Π the $x,\ x',\ 1-x-x',\ x_{\prime},\ x_{\prime}',\ 1-x_{\prime}-x_{\prime}'$ by c & by c' in Φ, the integrals being taken from 1 to zero separately for the x & x', as for $x_{\prime}$ & $x_{\prime}'$.

As for the third method, it is equally easy to see that it will suffice to take for the z the values of E, F, G, H which will be the same as in the first method. If one takes the second hypothesis of this third method, Π & Φ will have the same form as above; but if one takes the first, Π & Φ will be equal to $E+F$, & $G+H$, & in Π & Φ the z will be multiplied by the same terms as in the analogous formulas of the first article.

One sees that this method would be general for any number of claims whatever.

V.

We have sought until here to evaluate the claim after two consecutive observed transfers; one could evaluate it also after the observation of the products of the claim, proportionally to the total mass of this claim for many cantons. Suppose this mass reduced to unity; that $p', p'', \ldots p'''^{n}$ indicate the fractions paid annually on it; b', b'', $\ldots b'''^{n}$ the number of times that each fraction has been paid, x', x'', $\ldots x'''^{n}$ the probability that each fraction p will be paid each year earlier than another fraction. The formula

$$\frac{\int \left\{ x'^{b'} \cdots (1 - x' \cdots - x'''^{n-1})^{b'''^{n}} \cdot \frac{cp'x' + cp''x'' \cdots + cp'''^{n}(1 - x' \cdots - x'''^{n-1})}{1 - cx' \cdots - c(1 - x' \cdots - x'''^{n-1})} \partial x' \cdots \partial x'''^{n-1} \right\}^{n-1}}{\int \left\{ x'^{b'} \cdots (1 - x' \cdots - x'''^{n-1})^{b'''^{n}} \partial x' \cdots \partial x'''^{n-1} \right\}^{n-1}},$$

will express for each mass 1 the value of the claim; we will not stop ourselves to consider this hypothesis & many others similar which one could form; those which we have chosen above, & especially the first, if the observed intervals between the transfers differ only in a unity of years, or of three months, they appear to us to approach more to the truth than any of those which one would form according to other principles. Indeed, it is necessary, in the questions of this kind, to prefer in general the particular & individual observations, to the general observations which are already themselves only the mean values taken according to the common method to determine them.

We will end this Memoir, by observing that one would find easily some analogous formulas to these, which would be applied in the calculus to all the annuities on life, & would serve to resolve the questions of this kind with more precision than one has made until now.