MÉMOIRE
SUR
LE CALCUL DES PROBABILITÉS*

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FIRST PART

Reflections on the general rule which prescribes to take for the value of an uncertain event, the probability of that event, multiplied by the value of the event in itself.

We ourselves propose here, not to discuss the proofs on which this general rule has been supported, or the objections which one has made against it, but to analyze the rule in itself. Read 4 August 1784

We suppose that a certain number of events procure to a man the advantages a, a', a'', &c. & that their probabilities are e, e', e'', &c.; that other events cause him some losses p, p', p'', &c. & that their probabilities are c, c', c'', &c. the advantage which results from it will be expressed, according to this rule above, by

$$\begin{gathered} ea + e'a' + e''a'' + \&c \\ -cp - c'p' - c''p'' - \&c. \end{gathered}$$

If this value is zero, one says that no advantage results from it.

We suppose next two men, of whom the first has the probability e to obtain the advantage a, & the probability c to experience the loss p; & the second probability e' to obtain the advantage a', & the probability c' to experience the loss p'; $ae - cp$ will express the advantage of the first, & $a'e' - c'p'$ the advantage of the second. If $ae - cp = a'e' - c'p'$, one says that the advantages of these two men are equal.

We suppose finally that two players have some opposing interests, & that e & e' being the probabilities of the two events, the first makes the first of the two players win a, & consequently makes the other lose the same value a; while to the contrary the second event makes the second win a', and the first lose consequently a'.

The advantage of the first will be expressed by $ea - e'a'$, & that of the second by $e'a' - ea$; & if $e'a' = ea$, one says that they play an equal game.

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Now, it is easy to conclude from this statement, that the rule does not give the real value of the advantage, but gives only a mean value of it. For example, if I have the probability $\frac{1}{3}$ to win 2, & $\frac{2}{3}$ to win 1, my advantage will be expressed by $\frac{4}{3}$; now, it is clear that I can obtain 1 or 2, but never $1+\frac{1}{3}$. If I have the probability $\frac{1}{3}$ of an advantage 2, & if another has the probability $\frac{2}{3}$ of an advantage 1, these two advantages are equally expressed by $\frac{2}{3}$, & yet I can win 2 or nothing, & the other can win 1 or nothing; our state consequently is not the same. If next I have the probability $\frac{1}{3}$ to win 2 against my adversary, & if he has the probability $\frac{2}{3}$ to win 1, one says that we play in an equal game, & our strength is not however the same, since I can win 2 against him, & since he can win only 1 against me.

This rule must therefore hold only in the case where it is acceptable to take the mean value; & it is necessary to seek now if then this mean value must be taken in the manner that the rule teaches it.

The real value of the advantage or of the disadvantage of an individual, is expressed by each particular advantage that he can experience, & by each loss that he can fear, each of these events having a certain probability; each advantage, each loss conserves then its proper probability, instead that one regards the mean value of these advantages & of these losses, as a value that one could be sure to obtain.

One can propose it under two different hypotheses, to substitute the mean value to the real value: in the first, this substitution takes place voluntarily; in the second, we can regard it as forced. For example, if I have a probability $\frac{1}{10}$ to win 100, & a probability $\frac{9}{10}$ to win 10, I can voluntarily exchange this expectation against a certain sum; or else, if this expectation is the result of a part of a commenced game, that I am forced to quit, one can demand what sum I have right to obtain in order to be compensated of the expectation which I abandon.

If on the contrary one proposes to me to expose myself to certain risks, & at the same time to the expectation of certain advantages, I can require a certain sum before exposing myself to these risks; or else one can require one of them from me, if the advantages are judged superior to the risks.

In all these cases, it is a certain sum which one exchanges against an uncertain sum, or reciprocally; or two uncertain sums, unequal & unequally probable, that one exchanges between them; this last case can even be related to the first, because one can not compare these two unequal & unequally probable sums, which one supposes could be exchanged between them, only by reducing them both to a mean value regarded as certain. This exchange can be counted freely made, or should, if it is necessary, be regulated according to justice.

We examine separately these two hypotheses.

In the first, where the exchange is supposed voluntary, one must regard it as all the other agreements to buy, where the things which one exchanges are not rigorously equals between them, since it is necessary that there is for each of them who exchange, a motive of preference; however one can not regard as arbitraries the conditions of the exchange, because there exists between the exchanged things a kind of equality, a mean value which forms that which one calls the *price of things*.

Likewise, although here the one who prefers a certain value to a certain combination of uncertain values, & reciprocally, is to choose among some things of different nature, which one can not regard as equals, & that he has necessarily a motive of preference:

however the determination of the mean value which one must regard as representing, in each particular case, a system of uncertain values, more or less probable, must not be regarded as arbitrary.

In the ordinary exchanges, competition, the relation of the reciprocal needs establishes a common price. Here the mean value must be established according to this principle, that the one who exchanges a certain value against an uncertain, or reciprocally, & the one who accepts the exchange, finds neither the one nor the other in this change any advantage independent of the motive of particular suitability which determines the preference, so that in a great number of similar exchanges, there exists between the two parties the greatest possible equality, either that one considers the system of all the exchanges made by preferring the certain sum to the uncertain expectations, or that one opposes between them two contrary systems, but of which each is formed from exchanges of different nature.

Now, one will obtain this greater possible equality if one fulfills first the following two conditions: 1.° that the case where there will be neither loss, nor gain for any of the two systems, is the most probable of all in the series of events; 2.° that the probability to win or to lose for each of the two systems approaches indefinitely $\frac{1}{2}$ in measure as the number of events is multiplied.

Next one could demand that there is a probability always increasing with the number of events, that any of the two systems will have on the other only an advantage or a disadvantage, either less than a given quantity, or less than a quantity proportional to the greatest advantage or disadvantage possible. Now, one will find 1.° that as for the first two conditions, the law which prescribes to make the advantages in inverse ratio to the probabilities, or the stakes in direct ratio to the same probabilities, not only fulfills these conditions, but is the one alone which can fulfill them; 2.° that there is no law which can give a probability always increasing as the advantages or the losses will not exceed a given value; 3.° that the law above fulfills & is the only one which can fulfill the condition that the advantages or the losses will not exceed any proportional part whatever of the greatest advantage or greatest disadvantage possible.

This rule is therefore the only one which one can admit in general.

If now this exchange is forced, that is to say, that it is not at all made freely & according to the sole principle that there must not be between the two parties any advantage independently of the motive of preference which induces them to the exchange; then it is according to the laws of equity that it must be regulated, & these laws require that one seeks to fulfill the same conditions, that is to say, that there results the least possible inequality in the law which regulates the exchange; it is therefore still solely, by following the same law, that one can fulfill this condition.

There results from this what we just said, a remarkable difference between these two cases. Indeed, in the second where the agreement is forced, the law must always be followed; but in the first, if the kind of equality that this law establishes does not appear sufficient, there must result from it that, if one acts in the least with prudence, one will not wish to form the agreement. In the first case, one is induced according to the law, because one can only consider the total mass of similar agreements, & to seek to make so that there results from it the least possible inequality. In the second, if one wants to act with prudence, if the object is important, one must yield to the agreement only as much as one can envisage the possibility to establish between the two parties a sufficient equality.

This put, we consider two players, of whom the one A has an expectation e to win, & a risk $1-e$ to lose; & the other B an expectation $1-e$ to win, & a risk e to lose, & that the stake of A is to that of B as e is to $1-e$; in a way that by winning A will win $1-e$, & that by losing he will lose e each time a certain sum regarded as unity.

If $e < 1-e$ in measure as the number of trials will be multiplied, the probability that A will have to win will approach $\frac{1}{2}$, but it will remain always under it, & e can be small enough in order that, even for a very great number of trials, this probability is still very inferior to $\frac{1}{2}$, while the probability of loss that the same player will have, would always be quite above $\frac{1}{2}$.

In the same case, the probability to not lose beyond an n^{th} part of the total stake, will grow in favor of A whatever be n; but if n is very small, it will be necessary to suppose the game continued a very great number of times in order that this probability becomes great enough.

It is necessary to observe finally, that for the same player A, the probability of to not win beyond a certain portion of his total stake, increases at the same time as that to not lose beyond the same proportion.

It is likewise of all the cases that one could choose, so that in general the one of the two players who has the smallest probability, wins in the combination of a great number of trials on the side of the expectation to not lose, & loses with regard to the expectation of winning much, although to the contrary the one who has a great probability loses from the expectation to win & at the same time is exposed to a less risk to lose much.

This manner to consider the law which we examine, & which consists in regarding the value which results from it for the expectations & for the risks, as a proper mean value to restore the greatest possible equality between those who exchange between them a certain value & an uncertain expectation, or two uncertain expectations, &c. has appeared to us to be able to make disappear most of the difficulties that this rule has appeared to present in its application.

We are going to examine here some, & we will begin with the famous problem of Petersburg.

In this problem, one supposes that a player A must give to a player B a coin if he brings forth tails on the first toss, two if he brings it forth on the second, four if he brings it forth on the third, & thus in sequence; & one demands what is the value of the expectation of B, or what sum he must give to A in order to play an equal game. The rule of the calculus gives this sum equal to

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots=\frac{1}{2}(1+1+1\cdots)=\frac{1}{2}\times\frac{1}{0}.$$

A conclusion which appears so much the more absurd, as supposing this stake of B greater than any given quantity; one can have a probability as great as one will wish, as B will lose in this agreement.

But one can observe 1.° that the case which becomes the most probable, by supposing that one continues the game, the one where there is neither loss nor gain, can not take place here, at least if one does not suppose the game repeated an infinite number of times.

2.$^\circ$ That the probability of B to win, will no longer approach to be equal to $\frac{1}{2}$, & consequently to be equal to the probability of loss, but by supposing also the game repeated an infinite number of times; it begins even to be finite only at this term.

3.$^\circ$ That the probability to lose only a certain part of the total stake as we have seen should increase with the number of trials, is finite for B only by supposing infinite the number of times that the game is repeated, & that in this case this part of the stake is necessarily still an infinite quantity.

One sees therefore that the principle on which we have said that the general rule must be founded, the one to put the greatest possible equality between two essentially different states, can have no place here, since this equality would require that one embraces the combination of an infinite number of games, so that the limit which, in the ordinary problems is an infinite number of games, is necessarily here an infinity of the second order.

It is therefore not the rule which is in default, but the application of the rule to a case which one presents as real, & which however can not be, since it supposes the reality of an infinite sum, of an infinite number of trials in each game, & of an infinite number of games. Thus the problem must be considered not as a real case, but as the limit of the real questions of the same kind as one can have in view.

This explication however is not yet satisfactory. Indeed, one has remarked, with reason, that the rule would appear to be in default even when one would limit the number of possible trials, because the sum that B ought give to A under this hypothesis in order to play at an equal game, is yet such, if the number of trials is in the least great, that any reasonable man would risk to give it. However most of the solutions given to this question, are themselves limited to say that it was necessary to limit the number of trials, either because beyond a certain number, it was necessary to regard the probability as too small, or because it was necessary that this number was such that the wealth of A, or the sum that he reserves to this game, suffices to pay that which he must to B, if tails does not arrive at the last trial.

Such is therefore now the case which remains to us to examine, the one where the number of trials is fixed, & where the sum which B must give, & the probability which he has to win being finite, the problem becomes a real problem.

We suppose that each game is limited to n trials, & that one pays 1 if tails arrives the first trial, 2 if it arrives the second, 4 if it arrives the third, 8 if it arrives the fourth... 2^{n-1} if it arrives the n^{th}, & 2^n if it does not arrive at all. The probabilities will be

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \cdots \quad \frac{1}{2^{n-1}}, \quad \frac{1}{2^n},$$

& the stake of B must be

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots+\frac{1}{2}+1=\frac{n}{2}+1,$$

& we will find first that B will begin to win when tails will arrive at a trial p, such that $2^{p-1} > \frac{n}{2}+1$, or $n < 2^p-2$. If $n = 2^p-2$, then there is neither loss nor gain; but in the same case $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{p-1}}$, or $1-\frac{1}{2^{p-1}}$, expresses the probability that B has to lose.

We suppose, for example, $p = 4$ & $n = 2^p - 2 = 14$; the loss of B will be 8. We will have $\frac{7}{8}$ for the probability that A will win, $\frac{1}{16}$ for the probability that he will have neither loss nor gain, & $\frac{1}{16}$ for this that B will win. But also because $n = 14$, it will be possible that he wins 16376, in truth the probability of this gain will be only $\frac{1}{16384}$. On his side A will have a probability $\frac{15}{16}$ of not losing, but he would win only 7 in the most favorable case, & could lose up to 16376.

One sees therefore that there is a very great inequality between the positions of A & B, by considering only a single trial, & that not only there are some circumstances where neither one nor the other must wish to consent to change the state where they are before the game against the one who results from this agreement, but that this must take place nearly generally. If one considers next a sequence of games, then one will seek to determine, either the sum regarded as unity, or the number n of trials, in a way 1.° that the probability to win for A & for B approaches to equality, 2.° that one has a great enough probability that neither A nor B in a number m of games will lose beyond a value which is a given proportion with m.

The number of games being then determined by the condition to be such that it has a probability nearly equal to unity, or even certitude that the loss that A can have will not at all exceed his wealth, or the sum that one believes that he will wish or is able to set into the game.

The same assumption of certitude is the only rigorous, it is the sole way that B is not here at disadvantage. Indeed, we take a most simple case, the one where of 100 tickets, B chooses 1 of them, & gives 1 to A, on the condition that, if this ticket arrives, A will give to him 100, & we suppose that one plays 200 trials, the probability that A will win will be expressed by $\frac{40464}{100{,}000}$, that he will neither lose, nor gain by $\frac{27203}{100{,}000}$, and that B will win by $\frac{32333}{100{,}000}$, the probabilities to win for A & for B will be therefore here very nearly as 5 to 4, & consequently already neighboring to equality. In the same case, the probability for A to not lose beyond 500 will be $\frac{235}{100{,}000}$, a risk already very small.[1]

One sees therefore that provided that B has the expectation to be able to play 200 trials, there is established in the game a sort of equality. It is true that the established law can take place only by supposing that if A would lose 200 times, he would pay 200×100, or 20,000; but even though he would not have them, as the probability that A will not lose above 10,000, for example, is then nearly equal to unity, & that in the other very rare cases, B would win always 10,000; it is easy to see that even though A would pay only this sum, B would still consent to play this game, where he can expect to win 10,000 by risking only 200. B under this hypothesis would keep besides a probability $\frac{32{,}333}{100{,}000}$ to win against a probability $\frac{40{,}464}{100{,}000}$ to lose, a probability $\frac{18{,}145}{100{,}000}$ to win 100 against a probability $\frac{27{,}066}{100{,}000}$ to lose 100, & a probability $\frac{14{,}188}{100{,}000}$ to win 200 or more against a probability $\frac{13{,}398}{100{,}000}$ to lose 200. Thus, however B would not have the absolute certitude that A would pay all the possible loss, his state in regard to B would conserve still a sufficient equality.

[1] *Translator's note*: Condorcet has made an error here. Let X have a binomial distribution with parameters $n = 200$ and $p = .01$. $\Pr(\text{B wins } 100(k-2)) = \Pr(X = k)$ for $k = 0, 1, 2 \ldots 200$. For A to lose more than 500 requires B to win more than 500, namely, $\Pr(X \geq 8) = .001013$. Furthermore, each of the remaining probabilities in this and the following paragraph are slightly in error.

It is necessary however here to consider two quite distinct cases, the one where, for example, the two hundred trials above form a linked game, so that if A & B agree one time to play them, they are engaged to continue the number of trials; & under this hypothesis, the state of each player, & the kind of equality which subsists between them, & which can be regarded as sufficient, is expressed as we just said it.

But if A & B conserve the liberty to make at each trial the same convention, there is moreover an observation to make: since it is by considering at each time the system of future trials that A & B are determined to play, there results from it that they must regulate the stake regarded as unity, in a manner that at each trial they can envision as possible the number of trials necessary in order to establish a sufficient equality, that is to say that it is necessary that the wealth of each of the two, or the sum that one has cause to believe that he would wish to risk, can suffice to this number of trials; thus in order to conserve the necessary equality, the stake must change after a certain number of trials. In some of the possible combinations, that is to say, in those where the wealth of one of the two players is arrived to a value which obliges to this change, if one makes this diminution of the stake to enter in the calculus, one will see that there must result from it necessarily the possibility to play a much greater number of trials; whence there must result also between the players a greater equality; because this kind of equality consists in this that if one considers the sequence of future trials, one has a probability nearly equal for each of the players, to lose or to win, & a very great probability that the loss or the gain of any of the two, will not exceed a very small part of the total stake: now, in this case, the first condition holds as in the preceding, & the part of the total stake can even be, under this last hypothesis, regarded as a given quantity.

The manner in which we have considered the established rule, can explicate also two contradictory phenomena which are themselves presented in the applications of this rule in some real cases.

It happens equally, & if a reasonable man A will refuse to give a sum b for the probability n to win a sum $a > \frac{b}{n}$, & also if a reasonable man B will consent to give a sum b' for the probability n' to win a sum $a' < \frac{b'}{n'}$.

The first case takes place when b is a considerable sum with respect to the state of the fortune of A, or in itself, & if n is very small.

The second takes place to the contrary particularly when b' is a very small sum, & when n' is an extremely small quantity.

In the first case, although, if the game were supposed to be repeated a very great number of times, it was favorable to A, however he will refuse to play it; 1.° because he can not continue it a great enough number of times; 2.° because for a single trial he has a very great probability to lose his stake, & by hypothesis, to make a loss which inconveniences him, or which deprives him of agreeable enjoyments.

In the second, B agrees to play, because the small sum b' is a very moderate sum of which he does not regret the loss, & of which the expectation to win the considerable sum $\frac{b'}{n'}$, engages him to expose himself, even with disadvantage, to this loss regarded as light: this is here the case of the lotteries.

There are some games where the strength of the players is not equal, & where one gives advantage to a banker; as the banker is obliged to play a very considerable game, which requires some advances, & exposes to the possibility of enormous losses; which besides he is subject for the stakes, to be submitted, with certain limits, at the will of the punters; & that finally if he would have no advantage, he would have, especially when the number of punters is great, & when they play very nearly the same game, a very great probability to make only very little loss & gain, it has appeared necessary to accord to him an advantage which gave to him an assurance to win at length; & the punters have consented to buy at this price the pleasure to play, & the one to conduct their game at their whim up to a certain point.

One has observed among the games which depend altogether on chance & on good play, ones had only a very short duration, while others conserved their vogue a very longtime; one of the causes of this difference, is the way to combine in these games, the influence of chance & of good play, so that the difference in strength of the players, when it is small, alters not at all sensibly in the two or three games which one wishes to play in a day, the equality of the probability to win, as they could have among them some equal players. If one gives too much to chance, one takes off to these games a great part of their pleasure; if the chance influences too little, the difference of strength becomes too sensible, it humiliates the self-respect.

We will note finally, that in the enterprises where men expose themselves to a loss, in the view of a profit, it is necessary that the profit be greater than the one which follows the general rule, it establishes equality: indeed, as in general one is delivered not at all as in the game, by the appeal of the pleasure to play, or as in the lotteries, by the expectation to win much with a small stake, one can have motive to risk, only an advantage which, by envisioning a series of similar risks, produces an assurance great enough to win, & a probability nearly equal to certitude of no loss at all beyond a certain part of the stake.

These reflections have appeared to us proper to accommodate the rule established in the calculus of probabilities, with the sentiment & with the behavior of reasonable & prudent men, in most of the cases where this rule would appear at first glance to be contrary to it.

SECOND PART

Application of the analysis to this question: To determine the probability that a regular arrangement is the effect of an intention to produce it.

I.

I suppose that there are n possible combinations, & that one alone of them is regular. If a cause has had the intention to produce this combination, it has taken place necessarily, & its probability will be 1; if, to the contrary, it has been the effect of chance, its probability will be $\frac{1}{n}$. The probability that it is the effect of intention will be therefore $\frac{1}{1+\frac{1}{n}}$, or $\frac{n}{n+1}$, & the contrary probability $\frac{1}{n+1}$.

I suppose now that there are m regular combinations; the probability of one of these combinations, in the case where there would have been intention to produce it, would be 1, & in the case where there has been no intention to produce it, it would be $\frac{m}{n}$; the probability that there has been an intention will be therefore $\frac{n}{m+n}$, & that there has not been $\frac{m}{m+n}$.

If $m = n$, that is to say, if all the possible combinations are regular, then the two probabilities are equal.

We suppose, for example, that one finds the word *roma*, & that one demands what is the probability that this word has been written with intention. One will observe that of the twenty-four possible combinations, the following nine, *roma, ramo, omar, omra, oram, maro, mora, armo, amor*, form equally some words, which one has been able to write with intention; one will have therefore here $n = 24$, $m = 9$, thus the probability that this word has been written with intention, will be $\frac{24}{33}$, or $\frac{8}{11}$, & the contrary probability will be $\frac{9}{33}$ or $\frac{3}{11}$. If this were on the contrary the word *in* that one found written, as the two combinations *in* & *ni* give equally a word which has a sense, one will have then $n = m = 2$, & the probability that these letters have been placed with the intention to write a word, would be equal to $\frac{1}{2}$.

II.

But one must not limit oneself here to consider the absolutely regular combinations, since, these are not the only ones which indicate the intention of a cause; if, for example, one calls e the number of elements which enter into a combination, it can arrive either that one only of these elements being given, all the others dependent on it, according to a constant law, or else that this constant law depends on 2, on 3, on e' elements.

Now it is clear 1.° that the constant law does not exist less, although it is dependent on a great number of elements, & that thus all these combinations must enter into the number of regular combinations.

2.° That however they must not be regarded as being also certainly regulars.

Suppose that one casts the eyes on the two sequences of numbers

$$\begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10. \\ 1, & 3, & 2, & 1, & 7, & 13, & 23, & 44, & 87, & 167. \end{array}$$

of which the first is such that each term is equal to two times the one which precedes it less the preceding term, so that, two terms being known, all the others are; & the second is such that each term is the sum of the four preceding, so that, four terms being given, all the others are. It is clear that these two sequences are regular, that each Mathematician who will examine them, will see that they are both subject to a law; but it is sensible at the same time that, if one stops one of these sequences at the sixth term, for example, one will be rather carried to regard the first, as being regular, than the second, since in the first there will be four terms subject to a law, while there are only two of them in the second.

III.

In order to evaluate the ratio of these two probabilities, we will suppose that these two sequences are continued to infinity. As then there will be in both an infinite number of terms subject to the law, we will suppose that the probability will be equal; but we know only a certain number of terms subject to this law; we will have therefore the probabilities that one of these sequences will be regular rather than the other, equals to the probabilities that these sequences being continued to infinity, will remain subject to the same law.

Let therefore for one of these sequences e be the number of terms subject to a law, & e' the number corresponding for another sequence, & let one seek the probability that for a number q of terms following, the same law will continue to be observed. The first probability will be expressed by $\frac{e+1}{e+q+1}$, the second by $\frac{e'+1}{e'+q+1}$, & the ratio of the second to the first by $\frac{(e'+1)(e+q+1)}{(e+1)(e'+q+1)}$.

Let $q = \frac{1}{0}$, & e, e' some finite numbers, this ratio becomes $\frac{e'+1}{e+1}$. Thus in the preceding example, if one stopped at the sixth term, one will have $e = 4$, $e' = 2$, & the ratio will be $\frac{3}{5}$: if one stops at the tenth, one will have $e = 8$, $e' = 6$, & the ratio will be $\frac{7}{9}$.

If one supposes that e & e' are of the same order as q, the same ratio becomes $\frac{ee'+e'q}{ee'+eq}$, & if one supposes $e = q = 1$, it will be $\frac{2e'}{1+e'}$.

If consequently we have an infinite number of elements, which we supposed to form a regular combination, & at the same time if among the possible combinations, those which are absolutely regular, those where x elements alone are subject to a law, are in equal number, the probability that chance will give a regular combination, will be expressed by $\int \frac{2x}{1+x} \partial x$; therefore the probability that the existing regular combination will be the effect of an intention, will be expressed by $\frac{1}{1+\int \frac{2x}{1+x}\partial x}$, & that it is the work of chance by $\frac{\int \frac{2x}{1+x}\partial x}{1+\int \frac{2x}{1+x}\partial x}$.

Now $\int \frac{2x}{1+x}\partial x$ being taken from $x = 1$ to $x = 0$, is $2 - 2\ln 2$; the two probabilities will be therefore as $\frac{1}{3-2\ln 2}$ & $\frac{2-2\ln 2}{3-2\ln 2}$, or as $\frac{10{,}000{,}000}{16{,}137{,}056}$ & $\frac{6{,}137{,}056}{16{,}137{,}056}$. If there is in the observed combination only p terms subject to the law, the probability that it is regular will be $\frac{2p}{1+p}$, so that it will be necessary to take this number in place of unity; the two probabilities will be therefore

$$\frac{2p}{(4-2\ln 2)p + 2 - 2\ln 2}, \ \& \ \frac{(2-2\ln 2)p + 2 - 2\ln 2}{(4-2\ln 2)p + 2 - 2\ln 2};$$

whence there results that the first will become smaller that the second when $p < \frac{2-2\ln 2}{2\ln 2} < \frac{6{,}137{,}956}{13{,}862{,}944}$.

IV.

But this first assumption does not appear conformed to Nature, & there is no reason to suppose that the number of absolutely regular combinations, is equal to the one of the combinations where only one part of the elements is determined by the law; the most natural hypothesis appears to be that which consists in supposing that the number of regular combinations where x elements are subject to the law, & $1-x$ given arbitrarily is proportional to any number of combinations whatever of $\frac{1}{0}$ elements of which $\frac{x}{0}$ are of one nature, & $\frac{1-x}{0}$ of another nature, that is to say, $\frac{1}{\pi\sqrt{(x-x^2)}}$, π being the semi-circumference of the circle, & $\int \frac{dx}{\pi\sqrt{(x-x^2)}}$ being 1 when the integral is taken from $x = 1$ to $x = 0$.

The probability to obtain by chance a regular combination, will be therefore expressed in this case by $\int \frac{2x\,\partial x}{\pi(1+x)\sqrt{(x-x^2)}}$, this integral being taken from $x = 1$ to $x = 0$, a formula which becomes then $2 - \sqrt{2}$. The probability[2] of the first hypothesis will be therefore here $\frac{1}{3-\sqrt{2}}$ & that of the second $\frac{2-\sqrt{2}}{3-\sqrt{2}}$. If the observed combination is subject only for p elements to a certain regularity, then these two probabilities[3] will be, the first equal to $\frac{2p}{(4-\sqrt{2})p+2-\sqrt{2}}$, the second to $\frac{(2-\sqrt{2})p+2-\sqrt{2}}{(4-\sqrt{2})p+2-\sqrt{2}}$; therefore the first is the greater, as long as $p > \frac{2-\sqrt{2}}{\sqrt{2}}$; thus in the case of one entire regularity,[4] the probabilities will be $\frac{100,000}{158,569}$ & $\frac{58,569}{158,569}$; & the first probability will carry away the second, until p becomes smaller than $\frac{58,569}{141,421}$.

V.

If one part of the elements appears to depart from the design which one observes in the combination, three different cases can be presented.

1.° One can regard them as adherents to these regular combinations, but where one part of the elements solely is subject to a law, & in this case to apply the preceding solution, p being then the number of elements subject to the law: this case holds when one does not know the causes of this irregularity, & when the elements which depart from the order which one observes, do not prevent at all that it is certainly not noted in the rest.

2.° These elements of which the number is $1 - p$, can be regarded as being determined by a necessary law: in this second case, the probability will be expressed by $\frac{2p}{1+p}$, as before, the second will be by $\frac{\int \frac{2x\,\partial x}{(1+x)\sqrt{(x-x^2)}}}{\int \frac{\partial x}{\sqrt{(x-x^2)}}}$, the integral being taken from $x = p$ to $x = 0$; because in this case, the combinations which announce an intention, can appear only to p elements alone. Now,

$$\int \frac{\partial x}{\sqrt{(x-x^2)}} = A\,(\cos = 1 - 2p),$$

&

$$\int \frac{2x\,\partial x}{(1+x)\sqrt{(x-x^2)}} = 2A(\cos = 1-2p) + \sqrt{2}.A\left[\tan = \frac{(1+\sqrt{2}).\sqrt{p-p^2}}{1-(1+\sqrt{2})p}\right] - \sqrt{2}.A\left[\tan = \frac{(1-\sqrt{2}).\sqrt{p-p^2}}{1-(1-\sqrt{2})p}\right]$$

The first probability[5] will be therefore $\frac{2p}{1+p}$ or $\frac{2p'}{1+p'}$;if beyond the elements determined by a necessary law, there are others of them which are not in accord with the regular

[2] *Translator's note*: Here Condorcet computes $\frac{1}{1+(2-\sqrt{2})}$ for the first probability.

[3] *Translator's note:* Putting $r = \frac{2p}{1+p}$, the first probability is $\frac{r}{r+(2-\sqrt{2})}$.

[4] *Translator's note*: Put $p = 1$.

[5] *Translator's note*: $\int_0^p \frac{dx}{\sqrt{x-x^2}} = \arcsin(2p-1) + \frac{\pi}{2} = \arccos(1-2p)$. The second integral has

combination which one observes, & if p' alone are subject to it. The second will be

$$P = 2 + \frac{\sqrt{2}A\left[\tan = \frac{(1+\sqrt{2}).\sqrt{p-p^2}}{1-(1+\sqrt{2})p}\right]}{A(\cos = 1-2p)} - \frac{\sqrt{2}A\left[\tan = \frac{(1-\sqrt{2}).\sqrt{p-p^2}}{1-(1-\sqrt{2})p}\right]}{A(\cos = 1-2p)}$$

3. One can have a part of the elements which announce an intention, when the others announce a contrary intention. Let p be the number of elements which announce the first intention, p' the number of the others, & $p+p'=1$: one will suppose first that p' elements are determined by a necessary law; & one will have under this hypothesis for the probability of this first intention $\frac{2p}{1+p}$, & P for the contrary probability. One will suppose next the same thing for the second intention, & the probabilities will be $\frac{2p'}{1+p'}$, & P', P' being that which P becomes, by putting p' in the place of p, & one will have for the probability of the first intention,

$$\frac{\frac{2p}{1+p}}{\frac{2p}{1+p}+\frac{2p'}{1+p'}+P+P'},$$

for that of the second

$$\frac{\frac{2p'}{1+p'}}{\frac{2p}{1+p}+\frac{2p'}{1+p'}+P+P'}, \ \& \ \frac{P+P'}{\frac{2p}{1+p}+\frac{2p'}{1+p'}+P+P'}$$

for that which there is no intention.

If the two preceding cases are combined with this one, one will find easily, by the same principle, that they become then these different probabilities.

VI.

There remains to us now to examine the case where a part of the elements is unknown to us. Let a be the number of known elements, a' that of the elements subject to a regular order, $1-a+a'$ the number of those which can be subject to it, the probability that there will be x subject will be $\frac{a'}{x}$. Thus, the probability that there is an order, will be in this case,

$$\int\left(\frac{a'}{x}\cdot\frac{2x}{1+x}\right)\partial x \text{ divided by } \int\frac{a'}{x}\partial x,$$

the integral being taken from $x = 1-a+a'$ to $x = a'$; this probability[6] will be therefore

$$2\cdot\frac{\ln\frac{2-a+a'}{1-a}}{\ln\frac{1-a+a'}{a'}};$$

the value, in today's notation, $\arccos(1-2p) - \sqrt{2}\arctan\left(\frac{(3p-1)\sqrt{2}}{4\sqrt{p-p^2}}\right) - \frac{\pi\sqrt{2}}{2}$. Condorcet's value corresponds to a discontinuous antiderivative of the integral $\int_0^p \frac{2(x+2)}{(1+x)\sqrt{x-x^2}}dx$ and is thus incorrect.

[6] *Translator's note*: The denominator of the numerator in the displayed equation should be $1+a'$.

the value of the second probability will remain the same. For example, in the case where one considers no law as necessary, one will have, by making $a = \frac{1}{2}$, & $a' = \frac{1}{2}$, the first probability equal to $2^{\frac{\ln 4 - \ln 3}{\ln 2}}$, & the second to $2 - \sqrt{2}$, or the first[7] equal to $\frac{83{,}006}{141{,}575}$, & the second to $\frac{58{,}569}{141{,}575}$. If $a = a' = \frac{1}{4}$, the first[8] will be $2^{\frac{\ln 8 - \ln 4}{\ln 4}}$, & the second $2 - \sqrt{2}$, or $\frac{67{,}807}{126{,}376}$ & $\frac{58{,}569}{126{,}376}$.

VII.

If one knows b elements subject to one necessary law, & if the $1 - a$ elements, of which the order is unknown, can be subject to it, one will take in place of the formula P above,

$$\frac{\int \frac{b}{p} P \, \partial p}{\int \frac{b}{p} \, \partial p},$$

the integral being taken from $x = b$ to $x = 1 - a + b$.

VIII.

One must not regard as absolutely rigorous the preceding formulas, *n^os. VI & VII*. It would appear indeed that the probabilities $\frac{a'}{x}$, $\frac{b}{x}$ to have a number x of elements either regular, or determined by a necessary law, do not express the true probabilities, but that it would be necessary to substitute for the first,

$$\text{I.} \quad \frac{x^{x+\frac{1}{2}}.(1-x)^{1-x+\frac{1}{2}}}{(x-a')^{x-a'+\frac{1}{2}}.(1-a+a'-x)^{1-a+a'-x+\frac{1}{2}}},$$

& for the second

$$\text{II.} \quad \frac{x^{x+\frac{1}{2}}.(1-x)^{1-x+\frac{1}{2}}}{(x-b)^{x-b+\frac{1}{2}}.(1-a+b-x)^{1-a+b-x+\frac{1}{2}}}.$$

But by paying attention to divide next the integrals by those of these functions I & II, multiplied by dx, the integrals being taken from $x = a'$ or $x = b$, to $x = 1 - a + a'$, or $x = 1 - a + b$.

However, as the combinations non-regular or not being subject to any law, are not really such, but only appear so to us, the first method which sets aside these combinations, & which determines uniquely the probability according to the others which we know for regulars & subject to a law, is perhaps preferable, & the probability that it gives more nearly to the true probability.

Indeed, the elements which do not offer to us regularity, those which appear independent of a necessary law, can be regularly arranged among them or directed by a necessary law, without that their order or their regularity strike us; it appears therefore natural to regard these elements rather as nulls, relative to that which must happen to the unknown elements, than as forming a probability against the order of these elements or against the existence of a necessary law, according to which they are arranged.

[7] *Translator's note*: Since $2^{\frac{\ln 4 - \ln 3}{\ln 2}} = 0.830075$, $2 - \sqrt{2} = 0.585786$ and their sum is 1.41586 it is clear that Condorcet is forming these probabilities as their respective ratios.

[8] *Translator's note*: This should be $\frac{\ln 8 - \ln 5}{\ln 2} = 0.678072$.