Addition au Mémoire sur la détermination de l'orbite d'une planète, à aide de formules qui ne renferment que les dérivées du premier ordre des longitude et latitude géocentriques*

M. Augustin Cauchy

Comptes Rendus Hebd. Séances Acad. Sci. 25, 879–883.
OC I, 10 (388), 434–438.

We conserve the notations adopted in the Memoir of which there is question (*see* the session of 29 November). One will have first to resolve, with respect to the unknowns ς, $\varsigma_{\prime}$, the two simultaneous equations

$$(1) \qquad \Phi_{\prime} - \varsigma_{\prime}\mathfrak{R}_{\prime} = \Phi - \varsigma\mathfrak{R}, \quad \mathscr{R}_{\prime}^2 + \frac{1}{\varsigma_{\prime}^2} = \mathscr{R} + \frac{1}{\varsigma^2}.$$

[880] If, for more commodity, one puts

$$\mu = \frac{\varsigma + \varsigma_{\prime}}{2}, \quad \nu = \frac{\varsigma - \varsigma_{\prime}}{2},$$

one will have

$$\varsigma_{\prime} = \mu + \nu, \quad \varsigma = \mu - \nu, \quad \varsigma\varsigma_{\prime} = \mu^2 - \nu^2,$$

and the formulas (1) will give

$$(2) \qquad \Phi_{\prime} - \Phi = \mu(\mathfrak{R}_{\prime} - \mathfrak{R}) + \nu(\mathfrak{R}_{\prime} - \mathfrak{R}), \quad \mathscr{R}_{\prime}^2 - \mathscr{R}^2 = \frac{\mu\nu}{(\mu^2 - \nu^2)^2}.$$

If in the last of formulas (2) one substitutes the value of ν drawn from the first, the final equation that one will obtain will be of the fourth degree in μ. There is more: if one neglects ν^2 vis-a-vis to μ^2, the final equation will be of the third degree only and of the form

$$(3) \qquad \mu = \mathscr{A} + \mathscr{B}\mu^3.$$

It is besides easy to be assured that there would not be, under the relation of exactitude, any advantage to substitute the equation of the fourth degree to that of the third, the

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 10, 2010 difference between the two values that these two equations furnish being generally insensible.

The approximate values of ς, $\varsigma_{\prime}$ being known, one will know, hence, the approximate value r, and those of the constants $a, \lambda = \left(\frac{K}{a^3}\right)^{\frac{1}{2}}$ and H, of which the last will be little different from $\sqrt{Ka}$. We add that the formula

$$\sin(\phi - \eth - p) = \frac{\mathscr{R}}{r} \tag{4}$$

will furnish, at the end of time t, the value of the variable $p + \eth$ which will be little different from $c + \mathfrak{p} + \eth + \lambda t$, and, hence, an approximate value of the constant $c + \mathfrak{p} + \eth$. Therefore, if one puts, for brevity,

$$\gamma = c + \mathfrak{p} + \eth, \tag{5}$$

one will know already the approximate values of the constants a and γ. In order to obtain the more exact values of the same constants, and at the same time some approximate values of the constants $\varepsilon \sin c$, $\varepsilon \cos c$, it will suffice to apply the linear method to equation (4). Then, in fact, by neglecting the terms proportional to the square of ε and to its higher powers, one will find, in a first approximation,

$$A\delta a + \Gamma\delta\gamma + (E\sin c + F\cos c)\varepsilon = \sin(\phi - \gamma - \lambda t) - \frac{P}{a}, \tag{6}$$

[881] the values of A, Γ, E, F being

$$\begin{cases} A = -\dfrac{P}{a^2} - \dfrac{3\lambda t}{2a}\Gamma, & \Gamma = \cos(\phi - \gamma - \lambda t), \\ E = \Gamma\cos\lambda t + \cos(\phi - \gamma), & F = \Gamma\sin\lambda t + \sin(\phi - \gamma). \end{cases} \tag{7}$$

By aid of formula (6) and from four observations, one will determine the approximate values of the unknowns $\delta a, \delta\gamma, \varepsilon \sin c, \varepsilon \cos c$; consequently, the approximate values of ε, c. One will be able next to correct anew the values found of $a, c, \gamma, \varepsilon$, or, that which reverts to the same, the values of a, c, $\mathfrak{p} + \eth$, ε, by the linear method applied to equation (4).

It is good to observe that the value of p corresponding to $t = 0$ is found represented by the sum $c + \mathfrak{p} + \eth$, when one neglects the terms proportional to ε, and by the sum $c + \mathfrak{p} + \eth + 2\varepsilon \sin c$, when one neglects only the terms proportional to the square or to the higher powers of ε. It is easy to conclude that if one represented by γ the value of p corresponding to $t = 0$, formula (6) will continue to subsist, with this single modification, that the value of the coefficient E would be determined, no longer by the third of the formulas (9), but by the following:

$$E = \Gamma\cos\lambda t + \cos(\phi - \gamma) - 2\Gamma. \tag{8}$$

If, in equation (4) one substituted its derivative

$$\Phi - \varsigma\mathfrak{R} = \frac{H - \lambda a^2 \varepsilon\varsigma\mathscr{R}\sin\psi}{r^2}, \tag{9}$$

in which one has

$$\varsigma^2 = \frac{1}{r^2 - \mathscr{R}^2},$$

or, that which returns to the same, the formula

$$(10) \qquad (\Phi r^2 - H)^2 (r^2 - \mathscr{R}^2) - (\Re r^2 - \lambda a^2 \varepsilon \mathscr{R} \sin\psi)^2 = 0,$$

the linear equation that one would obtain in place of formula (6) would contain only the three unknowns

$$\delta a, \quad \varepsilon \sin e, \quad \varepsilon \cos e.$$

The preceding calculations determine neither the longitude ☊ of the ascendant node, nor the inclination ι, of which the cosine is supposed little different from unity. If, after having found the approximate values of r and of $p + ☊$, one wished to deduce those of the constants ☊, ι, it would suffice to operate in the following manner:

[882] First one will be able to draw the approximate value of ρ from the formula

$$(10) \qquad \rho + R\cos\chi = r\cos(\phi - p - ☊),$$

or better yet from formulas (18) on page 780,[1] next those of the coordinates x, y, z of the observed star, from the three formulas

$$(11) \qquad x = R\cos\varpi + \rho\sin\phi, \quad y = R\sin\varpi + \rho\sin\phi, \quad z = \rho\tan\theta.$$

Putting then

$$(12) \qquad \alpha = \sin ☊ \tan\iota, \quad \beta = \cos ☊ \tan\iota,$$

one will have, among α, β and x, y, z, the linear equation

$$(13) \qquad \alpha x - \beta y + z = 0,$$

which represents precisely the plane of the orbit. One will have therefore, by aid of this equation, and from two distinct observations, to determine approximately α, β, or, that which reverts to the same, ι and ☊.

As we have already remarked, the use of formula (6) supposes cos ι little different from unity, that which generally holds for the planets. But, after having drawn from formulas (1), (3), (11) and (13) some approximate values of a, α and β, one would be able, without recurring either to equation (6), or to the assumption under which it is applied, to deduce directly the corrections $\delta a, \delta\alpha, \delta\beta$ with the approximate values of the products ε sin c, ε cos c, from five observations and from the linear equation

$$(14) \qquad \delta a + \frac{s}{Sr}(x\delta\alpha - y\delta\beta)\sec\theta - a\varepsilon\cos(c + \lambda t) = r - a,$$

[1] See "Mémoire sur la détermination de l'orbite d'une planète à l'aide des formules qui ne renferment que les dérivées du premier ordre des longitude et latitude géocentriques." *Comptes Rendus Hebd. Séances Acad. Sci.* 25.

the values of x, y, r, s, S being determined by aid of the formulas

$$X = \cos\varpi\tan\theta - \beta\sin\chi, \quad Y = \sin\varpi\tan\theta - \alpha\sin\chi, \quad Z = (\beta\sin\varpi - \alpha\cos\varpi)\tan\theta,$$
$$S = \alpha\cos\phi - \beta\sin\phi + \tan\theta,$$
$$x = \frac{R}{S}X, \quad y = \frac{R}{S}Y, \quad z = \frac{R}{S}Z,$$
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tau = \frac{z}{\sin\theta}, \quad s = \tau + R\cos\theta\cos\chi.$$

Equation (14) is precisely that to which formula (3) is reduced on page 705,[2] when one equates to zero the first approximate value of r, and that one puts, consequently,

$$\mathscr{A} = 1, \quad \mathscr{C} = 0, \quad \mathscr{E} = -a\cos(c + \lambda t), \quad \delta\varepsilon = \varepsilon.$$

[883] We add that after having determined approximately a, γ, α, β, consequently $a, \mathfrak{p}, \iota$, ♉, one is able to make equations (1) from page 704 serve, combined with formula $\delta\varepsilon = \varepsilon$, to deduce from four or even from three observations the four corrections $\delta a, \delta\mathfrak{p}, \delta\iota, \delta$♉ with the approximate values of $\varepsilon\sin c$, $\varepsilon\cos c$.

[2]See “Mémoire sur la détermination et la correction des éléments de l’orbite d’un astre.“ *Comptes Rendus Hebd. Séances Acad. Sci.* 25.