Note sur l'application des formules établies dans les précédentes séances, la détermination des orbites des petites planètes*

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The formulas that I have given in the Memoir of 20 September, for the determination of the orbits of celestial bodies, have been applied, in this Memoir, to the calculation of the distances from Mercury to the earth and to the sun. It was important to show, by a new example, the advantages which the same formulas present, especially the new method of interpolation, when one applies them to some stars more distant from the sun, and especially to the small planets. Under this plan, I have sought the distances which separate, at the epoch of 12 July, the sun and the earth from the new planet of Mr. Hencke, by departing from seven positions of this planet, transmitted by Mr. Yvon Villarceau to Mr. Faye, who has had the accommodation to communicate them to me. By applying the method of interpolation to the seven observations at the same time, I have obtained some fourth differences which, being succeeded without any law, ought have probably depended on the errors of observation, and that one made effectively disappear by supposing each of these errors equal [532] or inferior to 3 sexagesimal seconds. There resulted from it that, in the developments of the longitude and latitude of the observed star in series ordered according to the ascending powers of time, one was able to neglect the terms proportional to the fourth power of time. This circumstance, very favorable to the precision of the calculations, had not been thus well established, if I myself was limited to make use of five observations alone; although then, as I am myself assured of it, one may be able to obtain, by limiting each development to three terms, some values sufficiently exact for the first two coefficients. I have wished to know also that which arrived if, by leaving aside the two extreme observations, one was limited to deduce from three of the five others the coefficients of time and of its square, and it is found that, in this case, the coefficient of the square of time was able to vary from the simple to the double, in the passage from an epoch to one neighboring epoch. These considerations are able to make better appreciate again the advantages of a method, which not only permit to make compete without much difficulty a rather considerable number of observations in the determination of the sought

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 8, 2010 coefficients, but which, moreover, serves to recognize in the numbers furnished by the observations of the probable errors of these same observations.

By operating as I just said, I have recognized that in the epoch of 12 July, the new planet of Mr. Hencke was separated from the sun and from the earth by some distances which represented sensibly the numbers 2.46 and 1.58. This conclusion accords besides with that which Mr. Yvon Villarceau has found (*see* the session of 13 September). I join here the result of my calculations.

The seven observations which I have taken for point of departure have been made at Berlin 5, 9, 11, 12, 13, 15 and 21 July. The epochs of these observations, counted by starting from the beginning of July; the longitudes and latitudes corresponding to the observed planet, or the values of ϕ and of θ; finally the heliocentric longitudes corresponding to the earth, or the values of ϖ, being those which the following table indicate:

Epoch of the observations		ϕ	θ	ϖ
July...	5.39515	$256^\circ\ 8'52.0''$	$18^\circ\ 41'\ 8.7''$	$283^\circ\ 9'\ 6.1''$
	9.40045	$255^\circ\ 23'43.1''$	$18^\circ\ 11'34.0''$	$286^\circ\ 58'15.9''$
	11.44948	$255^\circ\ 2'56.0''$	$17^\circ\ 55'36.2''$	$288^\circ\ 55'32.3''$
	12.47867	$254^\circ\ 53'17.9''$	$17^\circ\ 47'26.6''$	$289^\circ\ 54'27.1''$
	13.45512	$254^\circ\ 44'24.1''$	$17^\circ\ 39'25.2''$	$290^\circ\ 50'21.3''$
	15.44091	$254^\circ\ 27'44.1''$	$17^\circ\ 23'12.1''$	$292^\circ\ 44'\ 3.0''$
	21.54714	$253^\circ\ 47'\ 1.6''$	$16^\circ\ 31'\ 8.4''$	$298^\circ\ 33'46.4''$

[533]Moreover, the logarithm of the distance from the earth to the sun, at the epoch of the mean observation indicated by the number 12.47867, was 0.0071186. In departing from these data, and by conserving the notations adopted in the Memoir of 20 September, by taking besides for the origin of time t the epoch of the middle observation, and by designating by Θ the Naperian logarithm of $\tan\theta$, I have deduced from my new method of interpolation the developments of ϕ,ϖ,Θ, or rather of $\Delta\phi,\Delta\varpi,\Delta\Theta$, and I have found:

$$\Delta\phi = -554.77''t + 24.764''\frac{t^2}{2} + 0.290''\frac{t^3}{6},$$
$$\Delta\varpi = 3434.75''t + 0.32392''\frac{t^2}{2} - 0.0161''\frac{t^3}{6},$$
$$\Delta\Theta = -0.008037t - 0.00016052\frac{t^2}{2} + 0.00000565\frac{t^3}{6}.$$

The values of $\Delta^4\phi$, $\Delta^4\Theta$, furnished by the calculation, was able to be attributed to the errors of observation. Thus, in particular, the values of $\Delta^4\phi$ were

$$0.3'',\quad 1.7'',\quad -5.9'',\quad 0'',\quad -3.9'',\quad -0.4'',\quad 0.3'',$$

and were reduced to the quantities

$$2.4'',\quad 3.8'',\quad -3.8'',\quad 2.1'',\quad -1.8'',\quad 1.7'',\quad 2.4'',$$

if one admitted an error of $2.1''$ in the value of ϕ furnished by the middle observation. Similarly, the values of $\Delta^4\theta$, furnished by the calculation were

$$-0.4'',\quad -4.0'',\quad -3.3'',\quad 0.'',\quad -7.5'',\quad 0.4'',\quad -0.4'',$$

and were reduced to the quantities

$$3.1'', \quad -0.5'', \quad 0.2'', \quad 3.5'', \quad -4.'', \quad 3.9'', \quad 3.1'',$$

if one admitted an error of 3.5″ on the value of θ furnished by the middle observation. This put, formulas (9), (10), from page 410, that is to say the equations

$$\frac{K}{r^3} = B - A^2 - D_t A, \tag{1}$$

$$\tau = \frac{K}{C\cos\theta}\left(\frac{1}{r^3} - \frac{1}{R^3}\right), \tag{2}$$

being applied to the determination of the distances r and τ, which, at the epoch of 12 July, separated the sun and the earth from the observed star, have given to me sensibly, the first

$$\frac{K}{r^3} = 0.0000,$$

[534] and the second,

$$\tau = 1.70.$$

By substituting this approximate value of τ into formula (16) of page 411, that is to say into the equation

$$r^2 = R^2 + 2k\tau + \tau^2, \tag{3}$$

I have drawn from it approximately

$$r = 2.576,$$

and then formula (2) has given to me

$$\tau = 1.598.$$

This last value of τ is already very near to the truth. Its logarithm

$$0.2036$$

is, to nearest ten-thousandth, the number obtained by Mr. Yvon Villarceau (*see* the session of 13 September).

I will make here an important remark. When one finds very nearly, as in the preceding example,

$$\frac{K}{r^3} = 0,$$

this indicates only that $\frac{K}{r^3}$ is very small, or of the order of quantities comparable to the errors of observation. In the same case, it is no longer the value of $\frac{1}{r^3}$, but the value of τ which is found determined approximately by an equation of the first degree, namely, by equation (2), reduced to the form

$$\tau = -\frac{K}{CR^3\cos\theta}. \tag{4}$$

We remark further that if one puts, for brevity,

$$s = \tau + k, \tag{5}$$

equations (3) and (2) will become

$$r^2 = s^2 + l^2, \quad s = h - \frac{g}{R^3} \tag{6}$$

[535] the constants g, h, l^2 being determined by the formulas

$$l^2 = R^2 - k^2, \quad g = -\frac{K}{C\cos\theta}. \quad h = k + \frac{g}{R^3}.$$

Besides, if one eliminates s among formulas (6), one will draw from it the equation

$$r^2 - l^2 - \left(h - \frac{g^2}{r^3}\right)^2 = 0. \tag{7}$$

Now, if one differentiates the first member of this last equation with respect to r, one will obtain, for the derived equation, the formula

$$r - \frac{3g}{r^4}\left(h - \frac{g^2}{r^3}\right) = 0,$$

that one is able to reduce to the trinomial equation

$$r^8 - 3ghr^3 + 3g^2 = 0. \tag{8}$$

This last admitting only two positive roots, it is easy to conclude from it that equation (7) offers only three positive roots. One of these three roots is $r = R$. The two others have for limits the roots of the trinomial equation, which are able to be easily calculated by aid of known methods. Hence also, in equation (7), the two positive and distinct roots of R, will be always easy to determine.

Moreover, if one is limited to determine, as we just said, the two positive roots of equation (7), distinct from R, one would not be able to say a priori which of these roots correspond to the observed star. One will not have this inconvenience to fear at all, by following the method exposed above, since, after having obtained, by aid of an equation of the first degree, a first approximate value of r or of τ, one will be able to deduce immediately, by aid of the linear method applied to the resolution of the two equations (6), some new values generally very exact of the two distances r and $s = \tau + k$, and, consequently, some new values of r and of τ. By operating thus for the planet of Mr. Hencke, I have found

$$r = 2.471, \quad s = 2.376, \quad \tau = 1.583.$$

I will terminate with a last remark. By examining very recently, according to the indication of Mr. Walz, an old volume of the *Annales de Mathématiques* (years 1811 and 1812), I have encountered a Memoir of Mr. Gergonne, in which he restored already the determination of the orbit of a star by an equation of the first degree. Only the unknown, in this [536] equation, was no longer the distance r or τ, but one of the rectangular coordinates of the observed star.