Mémoire sur la détermination des orbites des planètes et des comètes

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The new methods that I have given for the determination of the orbits and of the movements of the celestial bodies are not, it is important to remark of it, some purely speculative theories; and if, on the one hand, they are able to contribute to the progress of mathematical analysis, on the other, they offer some means to calculate what are applied usefully to the solution of the great problems of astronomy. Already, in previous Memoirs, I have deduced some numerical results from the method by which I had arrived to determine, in the theory of planetary movements, the perturbations of a higher order, and I have shown that one was able to resolve thus the great inequality discovered by Mr. Le Verrier in the movement of Pallas. I had promised to show also, by an example, the advantages that my formulas present for the determination of the orbits of the planets and of comets. By ceding today to the view of many scholars who demand the accomplishment of this promise, I am happy to think that I [402] will furnish thus to the astronomers, and, in particular, to my honorable colleagues of the Bureau of Longitudes, an easy method to fix with a great precision, the elements of the orbits of the small planets recently discovered by diverse astronomers. We enter now into some details.

In order to determine also exactly as one is able the elements of the orbit of a planet in a given epoch, by deducing them from astronomical observations, it is agreeable to resolve successively two problems quite distinct the one from the other. The first consists to develop the variable quantities, especially the longitude and the latitude of the planet, according to the powers of time measured starting from the given epoch. The second problem consists in substituting the coefficients of the first terms of the developments of which there is concern, into some simple formulas of which one is able easily to deduce the distances from the planet to the sun and to the earth, and, hence, the diverse elements of the sought orbit. The first problem is able to be easily resolved by aid of the method of interpolation which I have proposed in a Memoir published in 1833, and reprinted in the Journal of Mr. Liouville. As I have remarked, this method offers numerous advantages which permit arriving promptly and surely to the sought developments. In fact, it substitutes for the investigation of the diverse terms of the development of any one of the variable quantities the search of one certain kind of finite

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 17, 2010 differences of diverse orders, represented by some linear functions of the observed values of the variable, or of the differences already calculated. Now these linear functions are easy to form, seeing that in each of them each coefficient is reduced, excepting the sign, to unity; and besides they are precisely those which offer the least chances of possible errors. This is not all; the method of which there is question is able to make agree to the solution of the problem any number of observations of which the results are combined among them by way of addition and of subtraction only; and the calculation, far from being complicated, while one advances, becomes so much simpler, as it is applied to the search of finite differences of a higher order. We add that the method, furnishing itself the proof of the justice of the effected operations, does not permit to the calculator to commit the slightest error, without that it is perceived almost immediately, and that the calculation is arrested of itself at the instant where one attains the degree of exactitude to which one was able to hope to arrive. We remark finally that the diverse terms of the sought developments are able to be easily deduced from the finite differences of which we just spoke, and from the determination of certain numbers of which the values [403] depend uniquely on the epochs in which the observations have been made. If the calculator knows these epochs without knowing the observations themselves, he is able immediately to calculate the numbers of which there is question, and to achieve thus, a singular thing, the most laborious part of all his calculation.

The first terms of the developments of the variables being calculated, as we just said, and deduced from observations of which I will suppose the number equal or superior to four, one will know the derivatives of the first three orders of each variable quantity, differentiated with respect to time, or rather their values corresponding to the given period; and there will remain no more than to substitute these values into some simple formulas of which one is able easily to draw the elements of the orbit with the distances which separate the sun and the earth from the observed star. We remark besides that this last problem would not be completely determined, if the number of observations were not at least equal to four, two orbits distinct from one another being able to satisfy three given observations. The admitted hypothesis is therefore that which it was acceptable to adopt. Now, under this hypothesis, if one takes for unknowns the distances from the observed star to the earth and to the sun, one will determine easily the value of these two unknowns by aid of the very simple operations that I am going to indicate.

1°. A linear equation will determine immediately the cube of the distance r from the star to the sun, and, hence, this distance itself. But, as this first equation contains the derivatives of the third order, of which the values are determined with less precision than those of the derivatives of the first and of the second order, the calculated distance will not be able to be rigorously exact, and it will be acceptable to make a correction to it substitute of which we will speak in a little while.

2°. When one will have determined approximately, as we just said, the distance r from the star to the sun, a second linear equation, which contains the derivatives of the first and of the second order, will furnish an approximate value of the distance r' from the star to the earth. We add that this last distance will be determined with a greater precision, if one deduces it from the resolution of the right triangle which has for vertices the centers of the earth, of the sun and of the observed star; that is to say, in other terms, if one deduces it from the resolution of an equation of the second degree which contains only, with the sought distances, two constants relative to the position of the earth, and two angles furnished by observation.

3°. When one will have determined approximately, thus as we just explained, the values of the two unknowns r, r', then, in order to obtain some [404] more exact values, it will suffice to apply the method of linear or Newtonian approximation to the simultaneous resolution of the two equations of which the distance r' from the earth to the observed star is deduced. In fact, it is important to remark, the method of Newtonian approximation is applied, with the greatest facility, to the rigorous evaluation, not only of a single unknown determined by a single equation, but again of two, three ... unknowns, determined by two or three equations, when already one possessed some approximate values of these unknowns. Besides, the two equations to which one will apply the method of which there is question contains no derivative of the third order, but only derivatives of the first and of the second order, which, by any means, could not be banished entirely from the calculation. Therefore, by definition, if one applies the Newtonian method to these same equations, after having determined the approximate values of r and r' by aid of the equation of second degree, joined to the linear equation which contains the derivative of the third order, one will obtain without any groping, and by a very rapid calculation, the distances from the observed star to the sun and to the earth, with an exactness equal or even superior to that which the laborious calculations produce in which one makes use of three observations only.

In order to verify these conclusions by an example which may be able to render evidence to all eyes, I have applied the method that I just exposed to the calculation of the distances which separate Mercury from the sun and from the earth, 16 August 1842, at the instant of its passage to the meridian, by deducing the distances from single observations made 14, 15, 16, 17 and 18 August, at the Observatory of Paris. The values that I have obtained, by taking for unity the distance from the earth to the sun, are, very nearly, as one will see it, in this Memoir, 0.32 and 1.30. These values, which will be found very slightly modified if, as it is easy to do, one has regard to aberration and to parallax, effectively coincident, when one neglects the digit of the thousandth, with those which furnish the astronomical Tables.

I will make, in terminating, a last remark.

When one will apply the new method to some stars for which the perturbations of the elliptic or parabolic movement will become sensible only at the end of a considerable time, it will be very advantageous to make agree to the determination of the elements of the orbit a great number of observations, when even the epochs of these observations would be rather elongated from one another. One will obtain thus much more exact values of the elements of the orbits, without increasing much the work of the calculator; because there will be changes only the numbers furnished by the formulas of interpolation; [405] and as one has seen it, the linear functions of which these formulas assume the formation are composed simply of the diverse values of the variable quantities, or of their differences of diverse orders, combined among them by way of addition or of subtraction.

§ I. — *Method of interpolation.*

Let t be time counted starting from a fixed epoch. A function u of t, supposed continuous in the neighborhood of this epoch, will be developable, at least between certain limits, according to the ascending powers of t; and if one puts

$$u = a + bt + c\frac{t^2}{1.2} + d\frac{t^3}{1.2.3} + \cdots, \tag{1}$$

the coefficients $a, b, c, d, \ldots$ will represent the diverse values of the function and of its derivatives of diverse orders, at the epoch of which there is question. If, besides, of the observations made with care they furnish particular diverse values of u corresponding to some positive, null or negative values of t, one will be able easily to deduce by the method of interpolation that I have given in 1833 the values of the coefficients $a, b, c, \ldots$ We enter to this subject in some details.

We suppose, for greater convenience, that the epoch starting from which time is measured is that of one of the observations. The value of a of u will be furnished by this same observation; and if one puts $v = u - a$, one will have

$$v = b + c\frac{t^2}{2} + d\frac{t^3}{2.3} + \cdots, \tag{2}$$

We suppose besides that, $f(v)$ being a function of t which vanish with t, one designates by St the sum of the numerical values of t relative to the diverse observations; next by $Sf(t)$ the sum of the values corresponding to $f(t)$, each of these last values being taken with the sign $+$, or with the sign $-$, according as it corresponds to a positive or negative value of t; and we represent by $\Delta f(t)$ a kind of finite difference of $f(t)$, determined by the system of the two formulas

$$\alpha = \frac{t}{St} \tag{3}$$

$$\Delta f(t) = f(t) - \alpha S f(t). \tag{4}$$

If, in the second of these formulas, one replaces successively $f(t)$ by $\frac{t^2}{2}$ and by v, one will draw from it

$$\Delta\frac{t^2}{2} = \frac{t^2}{2} - \alpha S\frac{t^2}{2}, \tag{5}$$

$$\Delta v = v - \alpha S v. \tag{6}$$

[406] We suppose further that one designates by $S'\Delta\frac{t^2}{2}$ the sum of the numerical values of $\Delta\frac{t^2}{2}$, relative to the diverse observations; next by $S'\Delta f(t)$, the sum of the values corresponding to $\Delta f(t)$, each of these last values being taken with the sign $+$ or with the sign $-$, according as it corresponds to a positive or negative value of $\Delta\frac{t^2}{2}$; and we represent by $\Delta^2 f(t)$ a kind of finite difference of the second order, determined by the system of the two formulas

$$\beta = \frac{\Delta\frac{t^2}{2}}{S'\Delta\frac{t^2}{2}}, \tag{7}$$

$$(8) \qquad \Delta^2 f(t) = \Delta f(t) - \beta S' \Delta f(t).$$

Finally, we imagine that by continuing in that way, one determines successively the values of the variables $\alpha, \beta, \gamma, \ldots$, by aid of the formulas

$$(9) \qquad \alpha = \frac{t}{St}, \quad \beta = \frac{\Delta \frac{t^2}{2}}{S' \Delta \frac{t^2}{2}}, \quad \gamma = \frac{\Delta^2 \frac{t^3}{2.3}}{S'' \Delta^2 \frac{t^3}{2.3}}, \text{ etc.,}$$

each of the signs S, $S'.S''$, ... indicating the sum of the numerical values of the variable quantity which it precedes, and the finite differences

$$\Delta \frac{t^2}{2}, \quad \Delta^2 \frac{t^3}{2.3}, \ldots$$

being determined themselves by the equations

$$(10) \qquad \Delta \frac{t^2}{2} = \frac{t^2}{2} - \alpha S \frac{t^2}{2}, \quad \Delta^2 \frac{t^3}{2.3} = \Delta \frac{t^3}{2.3} - \beta S' \Delta \frac{t^3}{2.3}, \text{ etc.}$$

It is easy to see that the quantities Δv, $\Delta^2 v$, $\Delta^3 v$, ..., determined by the system of formulas

$$(11) \qquad \Delta v = v - \alpha S v, \quad \Delta^2 v = \Delta v - \beta S' \Delta v, \quad \Delta^3 v = \Delta^2 v - \gamma S'' \Delta^2 v, \text{ etc.,}$$

will represent some kinds of finite differences of diverse orders of the function v, which will all vanish, if $u = v + a$ is reduced to a linear function of t; all, with the exception of the first, if u is reduced to an entire function of t and of the second degree; all, with the exception of the first two, if u is reduced to an entire function of t and of the third degree, etc.

This principle being admitted, we suppose the diverse observations made at some epochs near enough from one another, in order that the time being counted starting from one of among them, some terms of the development of u, for example the first four or five terms, remain alone [407] sensible. Then the numerical calculations of the values of Δv, $\Delta^2 v$, $\Delta^3 v$, ... corresponding to the diverse operations, will not linger to furnish quantities which will be obviously null, and which will be able to be neglected, in regard to the degree of approximation that one is able to hope to attain, for example of the quantities which will be comparable to the errors of operation. Now it is clear that one ought to be stopped since then, without seeking to push further the calculation of the finite differences

$$\Delta v, \quad \Delta^2 v, \quad \Delta^3 v, \ldots$$

It is clear also that by reducing to zero the first of these differences, which will be of the order of the quantities that one neglects, one will reduce the system of formulas (11) to a small number of equations which, joined to formulas (9) and to the equation

$$(12) \qquad u = a + v,$$

will furnish immediately the value of u developed into a series ordered according to the ascending powers of t.

It is important to observe that, if the calculator knows the epochs of the observations without knowing the observations themselves, he will be able immediately to deduce from formulas (9) and (10) the values of the variables $\alpha, \beta, \gamma, \ldots$ corresponding to the diverse observations, and, hence, the general values of these variables expressed as entire functions of t. We observe further that the exactitude of these first observations will be able to be easily established, seeing that by virtue of formulas (4) and (8), the values of Δt, $\Delta^2 t^2$, $\Delta^3 t^3$, ..., corresponding to each observation, should be rigorously null. We observe finally that, in formulas (9), (10), one is able, without inconvenience, to assume the numerical divisors, and to replace consequently these formulas by the equations

$$\alpha = \frac{t}{St}, \quad \beta = \frac{\Delta t^2}{S' \Delta t^2}, \quad \gamma = \frac{\Delta^2 t^3}{S'' \Delta^2 t^3}, \text{ etc.,} \tag{13}$$

$$\Delta t^2 = t^2 - \alpha S t^2, \quad \Delta^2 t^3 = \Delta t^3 - \beta S' \Delta t^3, \text{ etc.} \tag{14}$$

One will be able likewise, without altering the values of $\alpha, \beta, \gamma, \ldots$, to replace the diverse values of t by some quantities which are to them obviously proportional. This remark permits calculating very easily the values of $\alpha, \beta, \gamma, \ldots$, which would answer to some equidistant observations.

If, in order to fix the ideas, one wishes to deduce the development of u of five equidistant observations, the time being counted starting from the epoch of the middle observation, one will be able to replace the five values of t by the five terms of the arithmetic progression $-2, -1, 0, 1, 2$. Then the values [408] of the variable quantities t, t^2, t^3, t^4 and of their finite differences will vanish for the middle observation; while these same values, and those of α, β, γ, δ will be furnished, for the four other observations, by the following table:

Values of	t $-2; -1; 1; 2$	t^2 $4; 1; 1; 4$	t^3 $-8; -1; 1; 8$	t^4 $16; 1; 1; 16$
	$St = 6$	$St^2 = 0$	$St^3 = 18$	$St^4 = 0$
Values of α	$-\frac{1}{3}; -\frac{1}{6}; \frac{1}{6}; \frac{1}{3}$			
Values of	αSt $-2; -1; 1; 2$	αSt^2 $0; 0; 0; 0$	αSt^3 $-6; -3; 3; 6$	αSt^4 $0; 0; 0; 0$
Values of	Δt $0; 0; 0; 0$	Δt^2 $4; 1; 1; 4$	Δt^3 $-2; 2; -2; 2$	Δt^4 $16; 1; 1; 16$
		$S' \Delta t^2 = 10$	$S' \Delta t^3 = 0$	$S' \Delta t^4 = 34$
Values of β	....	$0,4; 0,1; 0,1; 0,4$		
Values of	....	$\beta S' \Delta t^2$ $4; 1; 1; 4$	$\beta S' \Delta t^3$ $0; 0; 0; 0$	$\beta S' \Delta t^4$ $13.6; 3.4; 3.4; 13.6$
Values of	....	$\Delta^2 t^2$ $0; 0; 0; 0$	$\Delta^2 t^3$ $-2; 2; -2; 2$	$\Delta^2 t^4$ $2.4; -2.4; -2.4; 2.4$
Values of	....	....	$S'' \Delta^2 t^3 = 8$	$S'' \Delta^2 t^4 = 0$
Values of γ	....	....	$-\frac{1}{4}; \frac{1}{4}; -\frac{1}{4}; \frac{1}{4}$	
Values of	....	....	$\gamma S'' \Delta^2 t^3$ $-2; 2; -2; 2$	$\gamma S'' \Delta^2 t^4$ $0; 0; 0; 0$
Values of	....	....	$\Delta^2 t^3$ $0; 0; 0; 0$	$\Delta^2 t^4$ $2.4; -2.4; -2.4; 2.4$
Values of	....	....	....	$S''' \Delta^3 t^4 = 9.6$
				$\frac{1}{4}; -\frac{1}{4}; -\frac{1}{4}; \frac{1}{4}$

[409] After having thus determined the particular values of $\alpha, \beta, \gamma, \delta$, relative to each observation, it will suffice, in order to obtain the general values of the same variables expressed as function of t, to recur to formulas (13) and (14), from which one will draw

$$t = 6\alpha, \quad t^2 = 10\beta, \quad t^3 = 18\alpha + 8\gamma, \quad t^4 = 34\beta + 9.6\delta,$$

and, hence,

$$\alpha = \frac{t}{6}, \quad \beta = \frac{t^2}{10}, \quad \gamma = \frac{t^3 - 3t}{8}, \quad \delta = \frac{5t^4 - 17t^2}{48}.$$

These last equations, joined to formulas (11) and (12), will be those which will serve to deduce, from five equidistant observations, the values of

$$D_t u, D_t^2 u, D_t^3 u, D_t^4 u$$

corresponding to the epoch of the middle observation, when the finite differences of u, of the fifth order, will be of the order of the quantities that one neglects, and comparable, for example, to the errors of observation.

We remark, in terminating this paragraph, that if the epochs of the observations are not rigorously, but sensibly equidistant, the values of $\alpha, \beta, \gamma, \delta, \ldots$ calculated as we just said, should only undergo very slight corrections, which it will be easy to determine.

§II — *Formulas for the determination of the orbits of planets and of comets.*

We take for origin of the coordinates the center of the sun, for plane of x, y the plane of the ecliptic, for semi-axes of the positive x and y the straight lines drawn from the origin to the first points of Aries and of Cancer, and for semi-axes of the positive z the perpendicular erected on the plane of the ecliptic, on the side of the north pole. Let, besides,

x, y, z	be the coordinates of the observed star;
r	the distance of this star to the sun;
τ	the distance of the same star to the earth;
ρ	the projection of this distance onto the plane of the ecliptic;
ϕ, θ	the geocentric longitude and latitude of the star;
x, y	the coordinates of the earth;
R	the distance of the earth to the sun;
ϖ	the longitude of the earth.

Finally, we take for unity of distance the semimajor axis of the terrestrial orbit, and we name K the attractive force of the sun at the unit distance. One will have, by supposing the observations contained in an interval of [410] time small enough in order that the perturbations remain insensible,

(1) $\quad x = \mathrm{x} + \tau\cos\phi\cos\theta, \quad y = \mathrm{y} + \tau\sin\phi\cos\theta, \quad z = \tau\sin\theta, \quad \rho = \tau\cos\theta,$

(2) $\quad \mathrm{x} = R\cos\varpi, \quad \mathrm{y} = R\sin\varpi, \quad \mathrm{z} = 0,$

and

(3) $$D_t^2 x + \frac{K}{r^3} x = 0, \quad D_t^2 y + \frac{K}{r^3} y = 0, \quad D_t^2 z + \frac{K}{r^3} z = 0,$$

(4) $$D_t^2 \mathrm{x} + \frac{K}{R^3} \mathrm{x} = 0, \quad D_t^2 \mathrm{y} + \frac{K}{R^3} \mathrm{y} = 0,$$

(and, by putting) $$\Theta = \tan\theta,$$

one will draw from the preceding formulas,

(5) $$D_t \rho = A\rho, \quad D_t^2 \rho + \frac{K}{r^3}\rho = B\rho, \quad \frac{K}{r^3} - \frac{K}{R^3} = C\rho,$$

the values of A, B, C being determined by the equations

(6) $$\begin{cases} C\mathrm{x} + [B - (D_t\phi)^2]\cos\phi - [D_i^2\phi + 2AD_t\phi]\sin\phi = 0, \\ C\mathrm{y} + [B - (D_t\phi)^2]\sin\phi - [D_i^2\phi + 2AD_t\phi]\cos\phi = 0; \end{cases}$$

(7) $$B + 2AD_t\Theta + D_t^2\Theta + (D_t\Theta)^2 = 0.$$

Besides, the first of the formulas (5) will give

(8) $$D_t^2\rho = (A^2 + D_t A)\rho;$$

and from that, joined to the two last of formulas (5), and to the equation $\rho = \tau\cos\theta$, one will draw

(9) $$\frac{K}{r^3} = B - A^2 - D_t A,$$

(10) $$\tau = \frac{K}{C\cos\theta}\left(\frac{1}{r^3} - \frac{1}{R^3}\right)$$

We add that if, for brevity, one puts $\psi = \phi - \varpi$, $v = \cot\psi$, one will have, by virtue of formulas (6) and (7),

(11) $$\begin{cases} A = -\dfrac{\mu}{2\nu}, \quad D_t A = -\dfrac{AD_t\nu + \frac{1}{2}D_t\mu}{\nu}, \\ B = (D_t\phi)^2 - vD_t^2\phi - 2AvD_t\phi \\ \quad = -D_t^2\Theta - (D_t\Theta)^2 - 2AD_t\Theta, \\ CR = \dfrac{(D_t\phi)^2 - B}{\cos\psi} = \dfrac{D_t^2\phi + 2AD_t\phi}{\sin\phi}, \end{cases}$$

the values of μ, ν, $D_t\mu$, $D_t\nu$ being [411]

$$\text{(12)} \qquad \begin{cases} \mu = D_t^2\Theta + (D_t\Theta)^2 + (D_t\phi)^2 - v D_t^2\phi, \\ \nu = D_t\Theta - v D_t\phi, \end{cases}$$

$$\text{(13)} \qquad \begin{cases} D_t\mu = D_t^3\Theta + 2D_t\Theta D_t^2\Theta + 2D_t\phi D_t^2\phi - v D_t^3\phi - D_t v D_t^2\phi, \\ D_t\nu = D_t^2\Theta - v D_t^2\phi - D_t v D_t\phi. \end{cases}$$

As for the value of $D_t v$, it will be evidently

$$\text{(14)} \qquad D_t v = -\frac{D_t\psi}{\sin^2\psi} = \frac{D_t\varpi - D_t\phi}{\sin^2\psi}.$$

We will observe finally, that by virtue of formulas (1) and (2), one will have

$$\text{(15)} \qquad r^2 = R^2 + 2R\tau\cos\theta\cos\psi + \tau^2,$$

or, that which reverts to the same,

$$\text{(16)} \qquad r^2 = R^2 + 2k\tau + \tau^2,$$

the value of k being

$$\text{(17)} \qquad k = R\cos\theta\cos\psi.$$

One could besides deduce directly formula (16) from this single consideration, that the three lengths r, τ, R are the three sides of the triangle of which the vertices coincide with the centers of the star observed from the earth and from the sun; and that, in this same triangle, the side R projected onto the base τ gives for projection the length represented by k.

One is able easily, from formulas which precede, to deduce without any groping, and with a very great exactitude, the distances from an observed star to the earth and to the sun, and, hence, the elements of its orbit. In order to arrive to it, one will draw first from equation (9) the distance r from the star to the sun. One will calculate next the distance τ from the star to the earth, by determining a first approximate value of τ, by aid of equation (10), next a second which will be generally more exact, by aid of equation (16), in which the constants R and k are immediately furnished by the movement of the earth, and by the givens of the observation.

After having obtained, as one just said, the approximate values of the distances τ and r, one will correct these values by having recourse anew, 1° to equation (10), which one will present under the form

$$\text{(18)} \qquad \frac{1}{r^3} = \frac{1}{R^3} + \frac{C\cos\theta}{K}\tau,$$

and from which one will draw a new value of r; 2° to equation (16), from [412] which one will draw a new value of τ. One will be able besides to obtain immediately, and usually, by aid of a single operation, some closely approximate values of r and of τ, by departing from the first calculated values, and by applying the method of linear or Newtonian correction to the system of the two equations (16) and (18). If one names δr, $\delta\tau$ the corrections that the values first found for r and τ should incur, one will have, by virtue of formulas (16) and (18),

$$(19)\qquad r\delta r-(k+\tau)d\tau=\mathfrak{a},\quad \frac{3}{r^4}\delta r+\frac{C\cos\theta}{K}\delta\tau=\mathfrak{b},$$

the values of $\mathfrak{a},\mathfrak{b}$ being given by the formulas

$$2\mathfrak{a}=R^2+2k\tau+\tau^2-r^2,\quad \mathfrak{b}=\frac{1}{r^3}-\frac{1}{R^3}-\frac{C\cos\theta}{K}\tau;$$

and, consequently, the values of $\delta\tau$ and δr will be

$$(20)\qquad \delta\tau=\frac{\mathfrak{b}r-\frac{3}{r^4}\mathfrak{a}}{\frac{Cr\cos\theta}{K}+\frac{3}{r^4}(k+\tau)},\quad \delta r=\frac{\mathfrak{a}-(k+\tau)d\tau}{r}.$$

§ III — *Application of the preceding formulas to the determination of the distances from Mercury to the earth and to the sun.*

Five observations, made at the Observatory of Paris, the 14, 15, 16, 17 and 18 August 1842, have furnished diverse values of the geocentric longitude and latitude of Mercury. The hours of these observations, completed starting from the midnight, were

$$11^{\mathrm{h}}\,27^{\mathrm{m}}\,17^{\mathrm{s}};\quad 11^{\mathrm{h}}\,31^{\mathrm{m}}\,32^{\mathrm{s}};\quad 11^{\mathrm{h}}\,35^{\mathrm{m}}\,46^{\mathrm{s}};\quad 11^{\mathrm{h}}\,39^{\mathrm{m}}\,58^{\mathrm{s}};\quad 11^{\mathrm{h}}\,44^{\mathrm{m}}\,8^{\mathrm{s}}.$$

The geocentric longitudes, or the values of ϕ furnished by the five observations, were

$$131^\circ\,25'43'',2;\quad 133^\circ\,25'42'',1;\quad 135^\circ\,26'22'',9;\quad 137^\circ\,27'37'',9;\quad 139^\circ\,29'8''.$$

The geocentric latitudes, or the values of θ furnished by the five observations were

$$1^\circ\,21'24'';\quad 1^\circ\,27'27'';\quad 1^\circ\,32'33'',5';\quad 1^\circ\,36'49'',9;\quad 1^\circ\,40'13'',8.$$

The heliocentric longitudes of the earth, or the values of ϖ corresponding to the hours of the five observations, were

$$-38^\circ\,47'53'',1;\quad -37^\circ\,50'1'',8;\quad -36^\circ\,52'9'',8;\quad -35^\circ\,54'16'',6;\quad -34^\circ\,56'22'',2.$$

[413] Finally, the logarithms of the distance from the earth to the sun, or the values of $\ln(R)$ corresponding to the same epochs, were

$$0.0054131;\quad 0.0053283;\quad 0.0052424;\quad 0.0051553;\quad 0.0050668.$$

Now, by carrying from these given, by taking besides for origin of time the middle epoch of the observation, and by putting, as in § II, $\Theta=\ln\tan\theta$, one draws from the method exposed in § I, the following values of ϕ, Θ, developed into series ordered according to the ascending powers of t:

$$\varpi = -36^\circ\ 52'9'', 8 + 57'42'', 7t + 1'', 1\,\frac{t^2}{2} + 0'', 2\frac{t^3}{6},$$
$$\phi = 135^\circ\ 26'22'', 9 + 2^\circ\ 0'39''t + 35'', 3\frac{t^2}{2} - 13'', 4\frac{t^3}{6} - 11'', 9\frac{t^4}{24},$$
$$\Theta = -3.614492 + 0.050475t - 0.011377\frac{t^2}{2} + 0.002117\frac{t^3}{6} - 0.002273\frac{t^4}{24}.$$

The coefficients of t, $\frac{t^2}{2}$, $\frac{t^3}{6}$, $\frac{t^4}{24}$, in these three formulas, are precisely the derivatives of diverse orders of the variables ϖ, ϕ, Θ. Thus, for example, the first formula gives $D_t\varpi = 57'42'', 7$, $D_t^2\varpi = 1'', 1$, $D_t^3\varpi = 0'', 2$. This put, there results from formula (9) of § II, that, 16 August 1842, the distance from Mercury to the sun was approximately 0.30. In the same epoch, the distance τ from Mercury to the earth was approximately, by virtue of formula (10) of § II, 1.67, and more exactly, by virtue of formula (16) of the same paragraph, 1.27. Now, by correcting the approximate values $r = 0.30$, $\tau = 1.27$, by aid of formulas (18) and (16) of § II, and by applying to these formulas the method of linear or Newtonian correction, one finds, in a first approximation, $r = 0.2196$, $\tau = 1.2911$. These values of r, τ, which are found slightly modified, when one takes account of aberration and of parallax, are, in fact, very little different from the true values of r and of τ, at the epoch of which there is question.