Extract from *Voyage Astronomique et Geographic dans l'Etat de l'Eglise...*

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1770, pp. 477–484, 501–512

299. We are going to give a list of the degrees on which one is able to form no reasonable doubt. The numbers of the first column mark the rank of each degree in particular, so that one is able to designate each by its rank; those of the second indicate the latitude of the corresponding degrees; & those of the third, the magnitude of the degrees, expressed in toises. The first degree is the one which has been measured in Lapland. I have drawn it from the work that Mr. *de Maupetuis* has published on this subject; but as one has not had regard at all to refraction, I have subtracted 16 toises from it, according as it is practiced today in regard to this degree. The eleven degrees which follow are drawn from the Book of Mr. *Cassini de Thury*, entitled *Méridienne vérifiée*; the thirteenth is the one that we have measured; the fourteenth is drawn from the works of Mr. *Bouguer* & Mr. *de la Condamine*, by taking a mean; the fifteenth, from a small pamphlet of Mr. l'Abbé *de la Caille* who has measured it.

	Latitudes	Degrees in toises		Latitudes	Degrees in toises
1	66° 20′	57422	9	45° 45′	57050
2	49 56	57084	10	45 43	57040
3	49 23	57074	11	44 53	57042
4	49 3	57069	12	43 31	57048
5	47 58	57071	13	43 1	56979
6	47 41	57057	14	0 0	56753
7	46 51	57055	15	−33 18	57037
8	46 35	57049			

To these degrees of meridian one is able to add the degree of the parallel that Mr. *Cassini de Thury* & Mr. l'Abbé *de la Caille* have found of 41618 toises, in the latitude of

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$43^\circ, 32'$.[1]

300. This will give already the means to make a number of comparisons, since

[1]One is able to add today yet other numbers which have relation to new measures made some in Italy by Fr. *Beccaria*, the others in Germany by Fr. *Liesganig*, & which are drawn from the manuscripts sent from *Turin* & from *Vienna*. The measures of Fr. *Liesganig* are actually imprinted; those of Fr. *Beccaria* will appear at the end of the collection of his works which are equally under press.

The latitude of *Mondavi* is $44^\circ, 23', 53''$; that of *Turin* $45^\circ, 4', 14''$; that of *Andrate* of $45^\circ, 31', 18''$.

One has found in toises of *Paris*, between *Mondavi* & *Turin*, 38680; between *Turin* & *Andrate* 26140 . *Andrate* is situated at the foot of a rather high mountain, & which on one side is the beginning of one of the grossest & the highest of the Alps, situated toward the north. From these measures & from these latitudes one draws the following table.

Between	Amplitude of the arc	Measure of the arc in toises of *Paris*	Value of the degree in toises of *Paris*
Mondovi & Turin	$40'\,41''$	38680	57075
Turin & Andrate	27 4	26140	57990
Mondovi & Andrate	1° 7 45	64820	57405

The latitude of *Vienna* in Austria is $48^\circ, 12'$. The toise of *Vienna* is to that of *Paris*, as 100000 to 102764. From this ratio, & from the measures, & from the intercepted arcs, one draws the following table.

Between	Amplitude of the arc	Measure of the arc in toises of *Vienna*	Value of the degree in toises of *Paris*
Vienna & Sobieschiz	$1^\circ\ 2'\,29.0'''$	61092.5	57082.3
Vienna & Brunn	0 58 53.5	57585.0	57090.8
Vienna & Gratz	1 8 24.8	66682.9	56909.6
Gratz & Varadin	0 45 49.9	45019.3	57351.3
Vienna & Varadin	1 54 16.5	111702.2	57717.7
Sobieschiz & Varadin	2 56 45.5	172794.7	57076.9

One sees on both sides the action of the mountains on the plumb-line of the sector (we have already paid attention to it in the note of n° 65, Book I). If one compares the first two degrees of Fr. *Beccaria*, which meet, the difference is 915 toises; a difference which surpasses that of the degrees measured under the equator and the polar circle, instead it must be only a very small number of toises. The plumb-line, attracted toward the north by the mountains of the Alps, has changed direction, & sent back from the side opposite the zenith that it indicates, by approaching it from the one of *Turin*: whence it is arrived that the arc intercepted by the two zeniths, is itself found too small & the degree too great. In order to evaluate the action of these mountains, we suppose in this place a degree which corresponds very nearly to this latitude, deduced from the preceding, & other degrees measured besides, namely of 57095 toises: it is more than one is able to give to it, & it must rather be less. One deduces from it the arc by this analogy: 57095 is to 57990, as the arc that one has found of $17'\,4'', = 1624''$, is to $1649''$ that one would have to have. The difference is $25''$, & it is also the difference by excess of the attraction of the mountains toward the north to *Andrate*, over their attraction to *Turin*. It is greater than that of the mountain of *Chimboraço* in America; but it is yet nothing, in regard to the magnitude of the mountains: this which proves, either that there is in these mountains even some great caverns, or that the Earth is much less dense toward the center than toward the surface.

The smaller mountains have an action less indeed, but which do not permit however being considerable enough. It is this which is seen in two degrees of Fr. *Leisganig*, one between *Gratz* & *Vienna*, the other between *Gratz* & *Varadin*. These degrees, although contiguous, different by 442 toises.

There has come to us yet recently another degree, measured in North America by Messers. *Mason* & *Dixon*, in the latitude of $39^\circ, 12'$, & which is found of 56888 toises of *Paris*. One sees it in the Philosophical Transactions, *year* 1768, *volume* 58, *page* 327. There is in this place a table of many degrees, of the number of which are the first degree of Fr. *Beccaria*, & a mean degree of Fr. *Liesganig*, such as they have themselves sent to the Royal Society, with a reduction by a small number of toises. One sees also two degrees chosen out of those which have been measured in France. The last two columns make known the Authors of the measure, & the year in which it has been done. We propose here this table of which we make use in the following notes.

any degrees of the meridian, taken two by two, determine the flattening of the Earth under the hypothesis of the ellipse of *Newton*. But one must not compare together two degrees too near one another, because the differences being then too small, a very slight error in the observations would produce a very considerable one in the result. If therefore of all the degrees of France one takes only the third, which is the one of Mr. *Picart*, for a latitude of 49°, 23′, a degree on which one has passed over again so many times, & with an exactitude so scrupulous; there will remain five degrees, namely the first which is the one of Lapland, the third which is a degree of France, & the last three which are those of Italy, of the Province of *Quito* & of the *Cape* of Good Hope. One is able first to examine, thus as we have done for the pendulum at the end of the first chapter, if the differences of the first degree of the meridian, which is nearest from the equator to the four other degrees, correspond to the versed sine of the double latitudes, or by how much they deviate. I have done it, & I have sought next the difference that the comparison of the degrees taken two by two gives, under this hypothesis of the proportionality with the versed sine: a difference which would be everywhere the same, if the differences of the degrees would always follow this proportion. Now the third of this difference, divided by the degree nearest the equator, determines the ellipticity, & I reduce this fraction, without changing the value of it, by a fraction of which the number is unity.

301. I will propose therefore two tables: the first has seven columns, of which the first contains by order the names of the degrees; the second, their latitude; the third, the half of the versed sine of a double latitude; the fourth, the number of toises of each degree; the fifth, the excess over the first degree of which the latitude $= 0$; the sixth, this same excess calculated under the hypothesis of the proportionality with the versed sines; the seventh, the error or the difference of the excess calculated to the excess observed. This first table will give us the means to construct the second, which has only three columns; of which the first contains the ranks of the combined degrees; the second, the excess that one concludes from it for a degree under the pole over a degree near the equator; the third, the fraction that gives the third of this excess divided by the first degree, that is the ellipticity. The excess of the degree of Mr. Abbé *de la Caille* over ours, proves the elongation of the figure. It is for this that in a second table I have marked with a negative sign its difference & its ellipticity compared to that of our degree, & that I have given the same sign to its error in the first table[2] that is here.

Degree in toises	Mean Latitude	Year of the measure	Authors of the measure
57422	66° 20′ north	1736 & 1737	Mr. *de Maupertuis.*
57074	49 23	1739 & 1740	Messers. *de Maupertuis* & *Cassini*.
57091	47 40	1768	Fr. *Liesganig*.
57028	45 0	1739 & 1740	Mr. *Cassini*.
57069	44° 44	1768	Fr. *Beccaria*.
56979	43 0	1752	Frs. *Boscovich* & *Maire*.
56888	39 12	1764 & 1768	Messers. *Masson* & *Maire*
56750	00 00	1736 & 1743	Messers. *de la Condamine* & *Bouguer*.
57037	33 18 merid.	1752	Mr. l'Abbé *de la Caille*.

[2]For this table we will substitute a greater one, drawn from the 9 degrees of our preceding problem, n° 299, & we will set here the degree of Mr. *de la Caille* in the rank which its latitude requires, as if this latitude were northern. The order of the degrees is indicated by the natural numbers, by beginning with the degree of the equator. We will make use of this table in the note on n° 303.

Degree	Latitude	$\frac{1}{2}$ versed sine for a radius of 10000	Number of toises	Difference of the first degree	Calculated difference	Error
From Quito	0° 0′	0	56751	0	0	0
From Cape of G. H.	33 18	2987	57037	286	240	−46
From Rome	42 59	4648	56979	228	372	144
From Paris	49 23	5762	57074	323	461	138
From Lapland	66 19	8386	57422	671	671	0

§302. In the last column of this table, one sees in how many the intermediate degrees deviate from the double ratio of the sine of the latitudes, or from the ratio of the versed sine of the double latitudes, supposing that the first & the last are correct. The calculated difference of the third & of the fourth degree being positive, that of the second is negative. When even there would arrive some slight change in the first and the last degree, the second would not deviate sensibly from this ratio. It is not so with the third and the fourth: the difference is already too sensible in order that one is able to accommodate them with the doubled ratio. We see now in the second table the excess of the last degree over the first proceeding from the comparison of the degrees taken two-by-two, & the ellipticity which results from it.

Order of the degrees	Value of the degrees	Latitude.	$\frac{1}{2}$ of the versed sign of a double latitude for the radius 10000	Difference observed in the first degree.	Calculated Difference.	Error
1	56750	00° 00′	0000	000	000	00
2	57037	33 18	3015	287	242	−45
3	56888	39 12	3995	138	320	192
4	56979	43 0	4651	229	373	144
5	57069	44 44	4954	319	397	78
6	57028	45 0	5000	278	401	123
7	57091	47 40	5465	341	438	97
8	57074	49 23	5762	324	462	138
9	57422	66 20	8389	672	672	0

This sequence, as one sees, is not at all regular; this which comes from the irregularity of texture of the internal parts of the Earth, & of the inequalities of the surface. The errors here above are assuredly not at all slight, no matter what some Authors say who attribute them to some observations of which they know not the degree of certitude, by never having made similar. It is easy to be convinced by the reader of this Book, that as little as an Observer is attentive, the errors committed in his observations will not produce in the degree a difference of 20 toises. We will be able to add also to the table of the n°, following a supplement in order to draw from it a mean ellipticity; but even this supplement will be found much better placed in the note on n° 303.

Degrees compared	Excess of the degree at the pole over the degree at the equator	Ellipticity
$1 \cdot 5$	800	$\frac{1}{213}$
$2 \cdot 5$	713	$\frac{1}{239}$
$3 \cdot 5$	1185	$\frac{1}{144}$
$4 \cdot 5$	1327	$\frac{1}{128}$
$1 \cdot 4$	542	$\frac{1}{314}$
$2 \cdot 4$	133	$\frac{1}{128}$
$3 \cdot 4$	853	$\frac{1}{200}$
$1 \cdot 3$	491	$\frac{1}{357}$
$2 \cdot 3$	-350	$-\frac{1}{486}$
$1 \cdot 2$	957	$\frac{1}{78}$

§303. One will be able also to compare the degree of the parallel measured by Messers. *Cassini de Thury* & *de la Calle* with those that one will wish of these degrees, by the problem of n. 280. but a degree of the parallel is not able to be measured with a sufficient precision. Now one sees by this table what is the irregularity of the degrees, since they give combinations so different. If one takes a mean among these ten combinations, the third of the mean excess will be 222, which gives for ellipticity $\frac{1}{133}$. But if one rejects the sixth & the ninth which are so different from the others, & of which the degrees are little deviated among them, the mean will be 286, & the ellipticity $\frac{1}{198}$. But this mean even differs yet much from many from among these eight determinations.

NOTE

Pour la fin du N^o. 303, Liv. V.

One must draw a certain mean ellipticity from all the degrees known through observations, compared among them, by having regard to the ratio that their differences must have, & to the rules of probability touching the correction that it is acceptable to make to them in order to reduce them to this ratio. Fr. *Boscovich* has done it in another work by means of a very curious method, & which is able to serve in many other cases. The result is exposed in it in an extract inserted into the acts of the Institute of *Bologna*. He develops it in his Supplements of the Philosophy in Latin verse, composed recently by Mr. Benoit *Stay*,[3] *volume* 2, *page* 420. We will insert here this article whole. Fr. *Boscovich* employs the numbers taken from the table which is at the end of page 407 of these Supplements: it is the same as that which he has set in this Book V, n^o. 301, & in which we have substituted one more ample in the note on this same section. We will apply next his method to this new table. Here is the statement in question.

"385. But in order to take this mean, such that it is not simply an arithmetic mean, but that it is bent by a certain law to the rules of the fortuitous combinations & of the calculus of probabilities; we will serve ourselves here with a problem that I have indicated toward the end of a Dissertation inserted in the acts of the Institute of *Bologna*, Here begins pp. 420–424 of Stay

[3] *Philosophiæ Recentioris a Benedicto Stay…versibus traditæ Libri X. cum adnotationibus, et supplementis P. Rogerii Josephi Boscovich…* Published in 3 volumes 1755, 1760 and 1792. Boscovich himself supplied commentary.

tome 4,[4] & where I myself am content to give the result of its solution. Here is the problem: *being given a certain number of degrees, to find the correction that it is necessary to make to each of them, by observing these three conditions: the first, that their differences are proportionals to the differences of the versed sine of a double latitude: the second, that the sum of the positive corrections is equal to the sum of the negatives: the third, that the sum of all the corrections, as many positive as negative, is the least possible, for the case where the first two conditions are fulfilled.* The first condition is required for the law of equilibrium, which demands an elliptic figure: the second, by a like degree of probability, for the deviations of the pendulum & the errors of the Observers, in the increase & diminution of the degrees: the third is necessary in order to be bring together the observations as much as it will be able; seeing especially that it is very probable that the deviations are quite small, as we have seen above; & that the scrupulous exactitude of the Observers does not permit suspecting some ever so little considerable errors in their observations.

"386. This problem has relation to the method *de maximis & minimis*; but one is not able to resolve it by the ordinary method of analysis. Because the algebraic expression distinguishes the positive quantities from the negative not at all, but it designates them by one same general value. One will have easily the value of the corrections that it is necessary to make in order to fulfill the first condition, by naming any two quantities, the one x, the other y, by means of which, & from the value of the degrees & from the versed sine, one will find any another corrected degree, of which the difference to the given degree, will give in x & y, & other known values, the analytic value of the correction, & the equation will be always of the first degree. In order to fulfill the second condition, it is necessary to equate the sum of all the values to zero: it is the sole position which is able to render the sum of the positives equal to that of the negatives. One will draw from this equation in x the value of y; & the substitution will give in x the sum of all the corrections. But this same sum, expressed by the analysis, will be a mixture of positive & negative quantities, & will not be variable at all; this which will be necessary in order to be able carried to a *maximum*; but it will be always $= 0$. Thus supposing $dx = 0$, one will have nothing: all the formula will vanish with the expectation of the calculator. But by means of simple Geometry, supported by mechanical, one comes easily to end of it, as one is going to see.

"387. Let (fig. 7. pl. I) AF be the diameter of a circle, & AE, AD, AC, AB the versed sine of the double latitudes, relative to the observed degrees: draw, from the points E, D, C, B, as if each degree had been observed under the equator, as also from the point A, which is in fact the place of the degree under the equator, the indefinite straight lines EE', DD', &c. perpendiculars to AF, & of which the segments Ee, Dd, Cc, Bb, Aa, taken on the same side, represent the degrees, so that one is able to note their extremities e, d, c, b, a.

" 388. One sees first that if one draws any straight line, as $A'H$, which encounters these straight lines at M, L, K, I, A', it will determine some degrees where the first condition will be fulfilled. For having drawn $A'F'$ parallel to AF, & which encounters these straight lines at E', D', C', B', the differences by excess of the degrees over the

[4]"De litteraria expeditione per pontificiam ditionem.", *De Bononiensi Scientiarum et Artium instituto atque academia commentarii*, IV (1757), pp. 353–396.

one of the equator, namely $E'M$, $D'L$, $C'K$, $B'I$, zero, will be proportional to the straight lines $A'E'$, $A'D'$, $A'C'$, $A'B'$, zero, that is to the versed sines AE, AD, AC, AB, zero. But the problem is yet undetermined in two places, since this straight line is able to be drawn to any distance, & since one is able to give to it such inclination as one will wish. Two degrees will be able already to determine it in some way; according to which it will determine itself, by its intersection with one of the parallel lines, which has ratio to any one latitude given, the degree which corresponds to it, according to the method proposed above (n^o. 292 of this Book V);[5] & this determination will give in x & y the values indicated in n^o. 386.

" 389. The second condition will determine a point on this straight line. The corrections will be eM, dL, cK, bI, aA', positives or negatives, according as the points e, d, c, b, a will be on this side or the other of $A'H$ with respect to AF. It will be necessary therefore, in regard to the second condition, that the sum of the corrections which are on this side, is equal to the sum of those which are on the other side; & it is that which one will have, if the straight line passes through the common center G of gravity of the points e, d, c, b, a, since by a well-known property of the center of gravity, the sum of the distances of all the points placed on one side, according to any direction, is equal to the sum of all those which are on the opposite side. Now these points being given, one has also their common center of gravity G. One has therefore a point of the straight line sought, determined by the second condition. This determination is equivalent to that value of y, that one must find, following n^o. 386, by the equation which supposes the sum of all the corrections $= 0$.

"390. The problem remains yet undetermined, since one is able to draw through this point an infinity of lines which will satisfy all the previous two conditions. The line determines therefore yet only a degree; it is the one which will be represented by GS perpendicular to AF, & which will correspond to a latitude of which the versed sine will be expressed by AS. Each other degree taken at will will determine this straight line, & hence likewise the other degrees. But it must be determined by the third condition, so that the sum of all the corrections (for on both sides they are always equal) is the least possible. For that we imagine a straight line $A'GH$, which departs from the position SGT, by turning to right or to left about the point G. First, & as much as the angle that it will form with it will be quite small, all the corrections aA, bI, cK, dL, eM will be enormous; next they will always go by diminishing, to that which the straight line

[5] n^o 292 From this that the degrees are in inverse ratio of the cubes of the perpendiculars lowered from the center onto the tangent, it is again easy to conclude that the increase of the degrees from the equator to the pole, will be very nearly in the same ratio as the square of the sine of the latitude, or as the versed sine of a double latitude; same ratio as the one of the diminution of the distance to the increase of gravity from the equator to the pole. For since the differences of the squares, of the cubes and of any powers, are quite small, they are in same ratio as the differences of the sides or of the roots. Thus the increases of the degrees will be in like ratio as the diminutions of the perpendiculars on the tangents. Now in an ellipse little different from a circle, the distance from the center to the point of contact is able to be taken for the perpendicular drawn from the center onto the tangent, even when the concern is of the difference from one perpendicular to another; for the perpendicular is the side of a right triangle of which this distance is the hypotenuse, & these two lines form an angle which corresponds to the ellipticity; whence it is easy to demonstrate by a method similar to that of which we ourselves are served above (n^o 232), that the difference of the perpendicular to the base, or the error that one will be able to commit, is infinitely small of the second order, & of no consequence. Therefore the diminution of the pependicular, & consequently the increase of the degree, is very nearly as the diminution of the distance, or in the same ratio as we have said.

attains some one of the points a, b, c, d, e: but since it will have passed, the correction which corresponds to this point will change directly from position, & will commence to increase, & it will always go by increasing, while those which have relation to the points not yet attained by the mobile straight line, will continue by decreasing. Now the sum of all the corrections will diminish to this that the sum of the differences relative to the increasing corrections, is greater than that of the differences of the decreasing; & it will be the least possible, since the former will cease to be less than the latter. But as soon as the sum of all the corrections will be the least possible, the sum of the positive corrections alone will be also the least possible, likewise the sum of the negatives alone, since these sums must be each the half of the total sum, because they are always equal to one another.

"391. Now the differences or changes of each correction, corresponding to the diverse changes of position of the mobile straight line, will be proportionals to the distances AS, BS, CS, DS, ES, whether they are increases or diminutions. Because these differences or changes will be on the bases of similar triangles, & of which the sum will be at G, & these bases will be comprehended between two positions of the straight lines GA', GI, GK, GL, GM; consequently they will be in ratio of these straight lines; that is, by the property of parallels, in ratio of AS, BS, CS, DS, ES. This is why if one observes in some order the mobile straight line must attain the points a, b, c, d, e, & if one adds together in the same order those of the straight lines AS, BS, CS, DS, ES, which correspond to these points; while this sum will be less than that of the half of the sum of all these straight lines taken together, or less than the sum of those which are on both sides of the point S (for the two sums taken the one to the right, the other to the left of this point, are equal to one another); the sum of the differences relative to the increasing corrections, will be yet less than that of the decreasing; the sum of all the corrections will go yet by diminishing, & this sum will be the least possible, when the sum of those of the straight lines AS, BS, CS, DS, ES which have relation to the points already encountered by the mobile straight line, will cease to be less than the half of the sum of all these lines, or that the sum of those which are on both sides of the point S.

"392. Now one will find easily the center of gravity G, & the order in which the mobile line encounters each point, & that by a numerical calculation which is nothing less than painful. This calculation consists in adding together the versed sines AE, AD, AC, AB, zero, & dividing the total by the number of the points in order to have AS, since the distance from the center of gravity to any plane Aa, is equal to the sum of the distance from all the points, divided by their number. Likewise if one divides the sum of all the degrees Ee, Dd, &c. by their number, one will have SG. It will even suffice to take the differences by excess of the degrees over the first, to make a sum from it that one will divide likewise by their number, & to add the quotient to the first degree. For if af is parallel to AF, & if it cuts the straight lines EE', DD', CC', SG, BB', AA' in R, Q, P, N, O; NG will be the sum of the excess Re, Qd, &c. divided by the number of the points.

"393. Now in order to find the order in which the points are encountered by the mobile straight line, one will draw through the point G a line parallel to AF, which will encounter the lines FF', EE', DD', CC', BB', AA' at Y, r, q, p, o at X; and one will see first in which of the angles SGY, YGT, TGX, XGS is found each point. For any point must be to left or to right of SGT, according as its versed sine is less or greater than AS; & below or above XGY, according as its degree will be smaller or larger than SG. One will not have difficulty any longer to find the tangent of the angle formed by GS or GT with the mobile line passing through any point. If it passes for example through the point c, one will have this analogy: re is to Gr, as the radius is to the tangent of the angle reG, or eGT, which will be consequently in ratio of $\frac{Gr}{re}$: it is thus of the others. The tangents of the small angles are smallest; & the points which correspond to some small angles are encountered rather by the mobile lines in the angles SGX, YGT; all to the contrary of that which arrives in the angles TGX, YGS. Therefore since Gr is the difference of the versed sine of the point e to the versed sine AS, & since re is the difference of the degree Ee to the degree SG; one will have the following rule: *divide for each point the difference of its versed sine to the versed sine AS by the difference of the degree which corresponds, to the degree SG; & that the quotients of the points which are found in two opposite angles at the summit, considered together, are ranked by order; by commencing with the smallest; that next one arranges also the quotients of the other points, placed in the two other angles, by commencing with the greatest. It is in this order that the mobile line will attain all these points, if it commences to be moved in the first two angles; & it will be the contrary if it would commence to be moved in the last two.* But unless there is need to recur to the calculus, the only construction, seeing that it is exact, will suffice ordinarily in order to understand with much more facility the order in which the points are encountered by the mobile line.

"394. By this means one has all that which is required for the corrections sought, & in order to have, even without their aid, the ellipticity. For the degree on which the mobile line reposes remains without correction, as one sees; consequently by means of this degree & of the degree SG, one will have by n$^{\text{o}}$. 348 (& by n$^{\text{o}}$. 301 of this Book V), all the other degrees; & hence even their difference to the degrees observed, that is the correction, & the total difference which will give the ellipticity that one seeks.

"395. Now one sees that the method is general for the correction of all the terms which must follow a given ratio; for by substituting this ratio to that of the versed sine, all returns to the same. But it is necessary to apply here the method to the degrees. We will serve ourselves with the values of the first table, n$^{\text{o}}$. 355 (& n$^{\text{o}}$. 301 of this Book), & in order to facilitate further the calculation, we will take the half of the versed sines for the entire versed sines. The values AB, AC, AD, AB are here the same as in the third column of this table, & by dividing their sum by 5, one has AS or $aN = 4356.6$. Ob, Pc, Qd, Re are the same values as those of the fifth column, of which the sum divided by 5 gives $NG = 301.6$, whence one draws the degree $SG = 56751 + 301.6 = 57052.6$, for one such latitude, that the half of the versed sine of a double latitude is 4356.6 for the radius 10000, that is for the latitude $41^\circ\,15'$: but we will make here no use of this calculation. The distances aN, ON, PN, QN, RN of the points a, b, c, d, e to the straight line SGT will be the differences of the first number $4356.6 = aN$, to the numbers of the third column; & consequently 4356.6, 1369.6, -291.4, -1405.4, -4029.4, the sum so many of the positives as the negatives, being 5726.2: & the distances aX, ba, cp, dq, er to the line XY will be the differences of the second $301.6 = NG$ to the numbers of the fifth column; & consequently 301.6, 15.6, 73.6, -21.4, -369.4. The distances which have some similar signs, are reported to the angles SGX, TGY; & those which have some different signs belonging to the other angles TGX, SGY. Thus the first are those of the points a, b, d, e; the point c is the only one of the second kind. Now if one divides the terms of the first sequence by those of the second, one will have, for the tangents of the angles with the line SGT, 14, 88, 4, 66, 11. Thus the four points which are found in the first angles, namely a, b, d, e, follow by beginning through the least angles the order of the numbers 11, 14, 66, 88, that is e, a, d, b; to which adding the point c, the straight line will encounter the points in this order e, a, d, b, c. The first distance from the first points e, namely 4029.4 is less than the sum 5726.2 of the two positives, or of the three negatives; but if one adds to it the distance from the following point, $a = 4356.6$, one will have 8386, which surpasses already this sum. Thus one will have the *minimum* that one seeks, when the straight line will attain the point a, & the position of this line aGV will correct all the degrees, with the exception of the single degree Aa at the equator, which will remain without correction.

"396. If the straight line moves in a contrary sense, it will encounter the points in a contrary order c, b, d, a, e, & one sees that in order to have a sum which surpasses 5726.2, it will be necessary to add together the first distances of the first four points, namely 291.4, 1369.6, 1405.4, 4356.6; so that this contrary movement will give yet the *minimum* to the encounter with the same point a.

"397. Having found the position required for the *minimum* sought, one will have first the ellipticity. For the position of the line will be here aGV, the degree of the equator remaining the same, this which is encountered fortunately in order to find all at once the total difference & the ellipticity. For one will have this analogie: $aN = 4356.6$ is to $NG = 301.6$, as $af = 10000$ to the total difference $fV = 692$: one will divide the degree 56751 by the third of this difference, & one will add 2 to the quotient, according to n$^{\text{o}}$ 350 (one will give it the demonstration below), in order to have the ellipticity $\frac{1}{248}$. The calculation gives for the five degrees a, b, c, d, e the following corrections: 0, -79.2, $+93.8$, $+75.9$, -90.5."

We apply this method to the numbers of the great table proposed in the preceding note on n$^{\text{o}}$ 301, by retaining Aa for the degree of the equator, Bb for any degree anterior to SG, Dd for any degree posterior. One will find the following values. Divide by 9, that is by the number of degrees, the sums of the fourth & fifth column, you will have $AS = aN = 4581.2$, $NG = 287.6$: thence the degree $SG = SN + NG = Aa + NG = 57037.6$; & since the half of the versed sine of a double latitude is $AS = 4581.2$, the latitude of this degree will be $42^\circ\ 36'$.

If one compares the number $4581.2 = AS$, with all the numbers of the fourth column which represent the lines AB, AD, by subtracting it from these numbers, one will have the first sequence of all the SB negatives, & of all the SD positives. Likewise if one compares the numbers $287.6 = NG$ with all the numbers of the fifth column which represent the lines Ob, Qd, one will have the second sequence of all the ob negatives, & of all the qd positives. We will name the first sequence A, & the second B. If one divides each term of the sequence A by the one which corresponds to it in the sequence B, one will have a new sequence C, which will represent the tangents of the angles formed by the lines Gb, Gd, with GS, GT; & the angles which correspond to these tangents, will be acute or obtuse, according as the value of the tangent will be positive or negative. Moreover it is not necessary at all to determine exactly these values; & unless the degrees are too neighboring, one nearly suffices, since there is question only of their relative magnitude in order to know the order where they must be placed.

	1	2	3	4	5	6	7	8	9
A	-4581.2	-1566.2	-386.2	69.8	372.8	418.8	883.8	1180.7	3807.8
B	-287.6	-0.6	-149.6	-58.6	31.4	-9.6	53.4	36.4	384.4
C	15.9	1610.	3.9	$-1.$	12.	-4.6	16.6	32.	10.

The numbers C make known the order in which the mobile line deviates from the position SGT, & passing through the positions $A'GH$, after a half-revolution about the point G, arrives to the extremities b, d of the degrees. One must begin through the positive quantities, from the smallest to the greatest; & continue next through the negatives, from the greatest to the smallest; & one will find the order in which the points are encountered by the mobile line, expressed by the following numbers: 3, 9, 5, 1, 7, 8, 2, 6, 4.

One will take in this order the sum of the numbers A, without having regard to the signs, until one attains a number equal, or greater than the half of the sum of all these numbers, which half is equal & to the sum of the positives, & to that of the negatives taken separately, since by the nature of the center of gravity they must be equals. Now the first is 6733.8, & the second 6733.6: they do not differ noticeably from the half of the sum, namely 6733.7; & this insensible difference comes from this that one has neglected some small fractions. If one takes in the sequence A the numbers 3, 9, 5, one arrives not yet to this half at all; but if one adds to them the number which corresponds to 1, one will have 9348.0 which surpasses it: consequently the sum of all the corrections will be the least possible, when the mobile straight line will attain the extremity of the first degree, which thence will remain the same without correction.

By this means one has already two degrees which must pass for exact, namely 56750 observed value for the latitude $= 0$, & 57037.6 for the latitude of $42^\circ\,36'$. From these degrees one is able already to deduce by the formula of n$^{\text{o}}$ 289, Book V, the ellipticity; but as one has here the first degree Aa, one will find it again, & with more facility, by this analogy: $AN = 4581.2 : NG = 287.6 :: af = 10000 : fV = 627.8$. This fourth term will be the difference of the degrees at the pole & at the equator; & if one divides the latter by the third of this difference, & if one adds 2 to the quotient, one will have 273 for the denominator of the ellipticity sought, which is found $\frac{1}{275}$ according to the theorem demonstrated by our Author in these same Supplements, n$^{\text{o}}$ 350, that we just cited in the passage drawn from these Supplements.

This ellipticity is again below $\frac{1}{248}$ which is that which would have first given the five degrees: but it approaches further $\frac{1}{335}$, an ellipticity required under the hypothesis of a spherical stone by the fraction $\frac{1}{176}$ (Book V. n. 251).

In order to correct all the degrees, it will suffice to seek the difference by excess from each degree to the degree under the equator, required by this determination; & to compare each of these differences with those of the fifth column, in the note on n$^{\text{o}}$ 301. Since $af = 10000$ must be to each line as aO, expressed by the numbers of the fourth column; as $fV = 627.8$ to Oi, of which the difference to Ob, expressed by the numbers of the fifth column, gives the sought correction; it will suffice to multiply the numbers of the fourth column by 627.8, & to divide the product by 10000; finally to subtract from the quotient the numbers of the fifth column: by this means one will have the following corrections 0, -97.7, $+112.8$, $+63.0$, -8.0, $+35.9$, $+2.1$, $+37.7$, -145.3. The sum of the positives is 251.5; that of the negatives 251.0, this which makes nearly equal sums; & the half of the total sum is around 251.2. Now the sum of these corrections, under the hypothesis that one observes the first two conditions, is a *minimum*; this which is evident by that same method of which one is served in order to find them: & one will be able again to be convinced, by trying to make such other substitution that one will wish; for it will give always a greater sum: it is this which one will make easily, by making an arbitrary correction in the first degree, & by determining by this degree also corrected, & by the one that one will find for the latitude of $42^\circ\ 36'$, all the other degrees, by means of the numbers of the fourth column, since the differences by excess of each degree over the first, must be in the ratio of these numbers.

It suffices to cast the eyes on the table of the note on n$^{\text{o}}$ 301, in order to perceive that the degree of Mr. Abbé *de la Caille* much disturbs the order of the degrees, since it surpasses the following two which are in a greater latitude. but on the other hand, this degree is in the southern hemisphere, instead as the other degrees are in our hemisphere: this which gives place to suspect that the two hemispheres do not resemble themselves for the figure. Thus it will be a propos to have regard in the calculation only to the 8 other degrees of this table.

By this means one will find the sums of the numbers of the fourth & of the fifth column, which, divided by 8, will give $aN = 4777.0$, $NG = 287.6$; whence one draws for the latitude of $43^\circ\ 43'$ the degree of $56750 + 287.6 = 57037.6$. Next one will have by the proposed method the numbers A, B, C, & the new sequence C will make known the order in which the mobile straight line attains the extremities of these 8 degrees, namely 3, 2, 4, 9, 7, 1, 8, 6: & if one takes in this same order the values of the new sequence A, one will arrive to the half of the total sum only after the addition of the sixth term, which corresponds to the first degree, that is to the degree under the equator, which must remain equally here under correction. Hence one has for the difference of the degrees under the equator & the pole 602.1, & for the ellipticity $\frac{1}{285}$, which approaches yet further the fraction $\frac{1}{335}$ required by that of the gravity under the hypothesis of a spherical stone. Now for the corrections to make to these 8 degrees, one will find 0, $+102.5$, 51.0, -20.7, $+13.0$, -12.0, $+22.9$, -166.9. The sum of the positives is 199.4; that of the negatives 199.6, they are nearly equal. The half of the total sum is 199.5, much less than the preceding, namely 251.2, since one does not take account of the correction of the degree of Mr. Abbé *de la Caille*; & that the other corrections differ very little from the preceding: the before last is increased by 20 toises; the others change only by 0 to 13 toises.

Of these eight degrees the second & the sixth have some much greater corrections than the others. If one omits further these two degrees, & if by the same method one takes a mean for the 6 remaining degrees, one will have always the first degree without correction; the degree 56998.5 for the latitude of $41^\circ\ 0'$; the total difference of the degrees under the equator and the pole 577.2; the ellipticity $\frac{1}{297}$, which approaches yet more the fraction $\frac{1}{335}$ required by that of gravity; finally for the corrections one will have 0, $+40.6$, -23.1, $+10.6$, -25.6, -9.6, corrections less than the first.

The quantities that one has found will serve to make known the absolute values of the axis & of the diameter of the equator: for in the ellipse the semi-axes are geometric mean proportionals between the radii of the curve taken alternately at their extremities; & consequently the degrees of the first enter those of the second; & when the terms differ little between them, one is able to substitute the arithmetic progression for the geometric. Therefore if to the degree under the equator (56750) one adds the third of its difference to the degree under the pole, or $\frac{1}{3} \times 577.2 = 192.4$, one will have for the degree of the semi-axis 56942.4, & if one adds again another third, one will have for the one of the semi-diameter of the equator 57134.6. It is for this that after having divided the degree of the meridian under the equator by the third of this difference, one adds 2 to the quotient in order to have the denominator of the ellipticity, or of the difference of the half-axes of the ellipse divided by the major axis. Now having the degrees, one will know easily the radii of the circles which correspond to them: one will find the semi-axis $= 3262560$; the semi-diameter of the equator $= 3273572$; their difference $= 11012$, a very small difference.

One will be served with the value that one just found of the degree of the equator, in order to reform that which expresses the ratio of the centrifugal force of the original gravity under the equator (n. 71. Book V.), namely $\frac{753}{216741}$, to which one has substituted the fraction $\frac{1}{288}$ which approaches much to it. But these numbers have been drawn from the degree of the equator that Mr. *Bouguer* had deduced by his theory, namely 57264 (n. 69); & thence since this degree is smaller, one must diminish the centrifugal force in double ratio of this degree. Thus this corrected ratio will be $\frac{753}{216741} \times \left(\frac{57134}{57264}\right)^2 = \frac{1}{289}$ very nearly.

All the combinations above give an ellipticity less than that which is required by the homogeneity of the fluid in the stone; whence it follows that the stone, supposed that it is spherical, must be denser than the fluid. The ratio of their density is able to be drawn from the formula of n$^{\text{o}}$ 199, Book V, in which in which $x = \frac{nr}{2m\left(1-\frac{3t}{5p}\right)}$; whence it follows that $\frac{t}{p} = \frac{5}{3}\left(1 - \frac{nr}{2mx}\right)$. Now $\frac{t}{p}$ is, in that very place, the sought ratio of the density of the fluid to that of the stone; & $\frac{m}{n}$ the ratio of the centrifugal force to gravity, that we just found $= \frac{1}{289}$; $\frac{x}{r}$ the ratio of the difference of the semi-diameter of the equator to the semi-axis; & consequently $\frac{r}{x}$ the denominator of the ellipticity, diminished by the unit, namely 296. If one likes better to take a mean between this denominator 297, & 335 which is the denominator required by gravity, namely 316, one will have $\frac{t}{p} = \frac{5}{3}\left(1 - \frac{315}{578}\right) = \frac{100}{131}$; that is that the density of the fluid will be to that of the stone nearly in the ratio of 3 to 4.

As the ellipticity that one draws from it is much smaller than that which one supposes commonly, there will be many calculations to reform; for example those which are based on the bulge of the earth at the equator in order to determine the precession of the equinoxes & the nutation of the axis; those which have relation to the parallaxes of the Moon, which depend on the ellipticity, & on the geographic distances of the places which depend on it also. In order to have all that with again so much more exactitude, it is to wish that one give us many other measures of degrees, finally to achieve in compensating & erasing totally the chance irregularities by the number of measures.

This was close to be set under press, when we received an extract of the measure of a new degree, made in Hungary by Fr. *Liesganig* who has achieved at the beginning of

February of this year (1770) in determining the value of it by calculation, & which will soon be printed. This measure is a confirmation of the irregularity of the degrees: & if one adds it to the others in order to take a mean, one will have an ellipticity near entirely conformed to that which gravity required under the hypothesis of a spherical stone. The amplitude of the arc, determined by the observations of two fixed stars, is found 1°, $1'$, $34.7''$, with so great accord, that one of the results has been greater, the other smaller, only in the tenth part of one second. In regard to the two measured bases to the two extremities of the meridian, one has been deduced from the other & by the chain of triangles by the calculus; & the base calculated differs only by one half-toise from the actual measure: whence it follows that one must not fear that there has even slid an error of 10 toises in the measure of a degree: & however this degree is found 56882.0 toises at *Paris*; & if one reduces it to the surface to the sea, one must yet remove a toise from it. Now its mean latitude is $45^\circ\ 57'$, & its longitude is near 4 degrees greater than that of *Gratz*. It has been measured in an immense plain, which was uninterrupted only be small inequalities, such as the waves on a calm sea: & it accords enough with the one which has been measured between *Vienna* & *Gratz* (n. 299, note), & which for a mean latitude a little greater, namely $47^\circ\ 38'$, is found 56909.6 or 29 toises greater than the one of Hungary. The one here agrees also to near 7 toises with the one of North America (n. 299 note), which has been equally measured with a very great exactitude in a vast plain, although it had near 6 degrees more latitude: the mean three degrees in the first table of this note, both surpass by more than 100 toises. All this seems to indicate an irregularity of texture in the Earth, even below its surface, & a figure of equilibrium so to speak undulating, as our Author had already formerly doubted; this which proves that two degrees are not able to suffice in order to determine the figure of the Earth: it is necessary on the contrary a great number of degrees measured in diverse countries, with this exactitude & those instruments which are today in use: & one must draw a certain mean from them by a sure method, & not on some simple prejudices made on the observations with arbitrary corrections, & greater in many regards than the methods invented in our day carry.

In order to find this mean, we will substitute into the preceding table, a the unique degree of Fr. *Leisganig*, that is at the mean degree of the arc which extends into Moravia, Austria & Stirie, three other degrees, namely the one of Hungary, the one is which between *Vienna* & *Gratz*, and the one which is between Vienna & *Sobieski*:[6] their mean latitudes are $45^\circ\ 57'$, $47^\circ\ 38'$, $49^\circ\ 13'$; their values in toises 56881, 56910, 57082: in this way one will have 11 degrees. The method above will give for the ellipticity $\frac{1}{311}$; & if one would omit the single degree of Lapland which differs too much from the others, especially from the degree of North America & the one of Bohemia, measured first by the English, second by Fr. *Leisganig*, one will have $\frac{1}{341}$. These two fractions do not deviate much from $\frac{1}{335}$ which gravity requires. It is demonstrated in this same work, that the inequalities which are near the surface troubles much more the measure of the degrees, than the length of the isochrone pendulum. Thus since these lengths follow near exactly the law of proportionality with the versed sine of a double latitude, & that this mean returns to the same; one will be able to take for ellipticity this same fraction $\frac{1}{335}$, & to draw the ratio of the density by means of the formula

[6] Poland

$\frac{t}{p} = \frac{5}{3}\left(1 - \frac{nr}{2mx}\right) = \frac{5}{3}\left(1 - \frac{334}{578}\right) = \frac{1220}{1734} = \frac{100}{142}$; this which makes nearly $\frac{2}{3}$. Thus the density of the fluid will be to that of the stone nearly in the ratio 2 to 3.

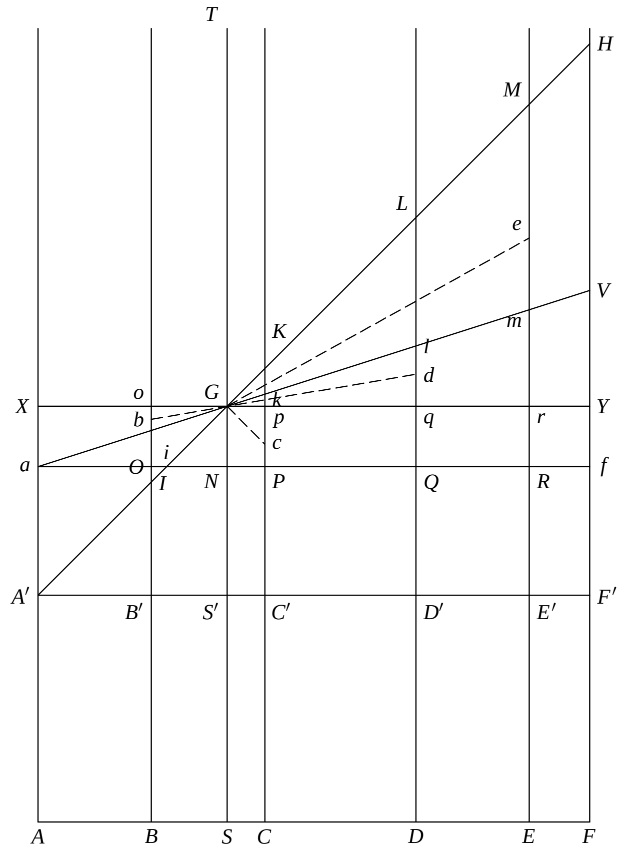

Figure. 7.