Boscovich's Example

Richard Pulskamp

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Here we examine the solution of Roger Boscovich for the ellipticity of the Earth as presented in the note published in Hugon's translation. There he gave a geometric method. Here we proceed by analysis.

Boscovich began with the measurements of arcs at five sites which are presented in the table taken from this same translation. The sites are listed in the first column. In the second is the latitude of each site. The third column is one-half the versed sine of the double latitude multiplied by 10000. Since versed sine $2\theta = 1 - \cos 2\theta = 2\sin^2\theta$, these values are the equal to that of the squared sine of the latitude. The fourth column is the observed length of one degree of arc at the corresponding latitude. Column five gives the excess of the length of each degree of arc over the length at the equator. Assuming the Earth is flattened at the poles, the length of a degree of meridian must increase as one departs from the equator. Column six is the computed excess of each degree of arc over the length. Finally, the last column presents the estimated error of the observed value as it is measured against the computed length.

DEGREE from	Latitude	$\frac{1}{2}$ versed sine for a radius of 10000	Number of toises	Difference from the first degree	Calculated difference	Error
QUITO	$0^\circ\, 0'$	0	56751	0	0	0
CAPE OF G. H.	33 18	2987	57037	286	240	−46
ROME	42 59	4648	56979	228	372	144
PARIS	49 23	5762	57074	323	461	138
LAPLAND	66 19	8386	57422	671	671	0

The Cape of Good Hope is treated as if it were in the northern hemisphere. Its entry in column 3 is erroneous and should be 3,014. This is due to using the latitude of $33^\circ\ 8'$ rather than $33^\circ\ 18''$ in the computation.

1 Boscovich's Model

What we immediately note from the table is this: Boscovich has concluded the measurements at Quito and in Lapland are without error.

Under the assumption that the Earth is an ellipsoid of rotation, it can be shown that the length of a short arc may be approximated as a linear function of the versed sine.

Let a_i denote the length of one degree of arc at latitude θ_i. We may write

$$a_i \approx a + b \sin^2 \theta_i \tag{1}$$

Since the observed lengths are approximations, when we substitute the data into this equation, we expect that equality will not hold. Therefore, let us put

$$e_i = a_i - a - b \sin^2 \theta_i \tag{2}$$

It is easy to verify from the table that Boscovich has found Equation 1 to be

$$a_i = 56751 + 800 \sin^2 \theta_i$$

in this case. For this result the sum of the absolute deviations from the line is $0 + |-46| + 144 + 138 + 0 = 328$. However, the sum of the errors does not vanish.

2 Boscovich's Proposal

In reality Boscovich wished to impose two conditions. These are

$$\sum e_i = 0 \tag{3}$$

and

$$\sum |e_i| \text{ is a minimum.} \tag{4}$$

He did not complete this program because the analytic techniques available to him led him to believe it was not possible. From Equation 3 and using $\overline{a}$ and $\overline{\sin^2 \theta}$ to denote the corresponding means, it follows that

$$a = \overline{a} - b \overline{\sin^2 \theta} \tag{5}$$

so that the constant a may be expressed in terms of b. Moreover with Equations 2 and 5, we may write

$$e_i = (a_i - \overline{a}) - b(\sin^2 \theta_i - \overline{\sin^2 \theta}) \tag{6}$$

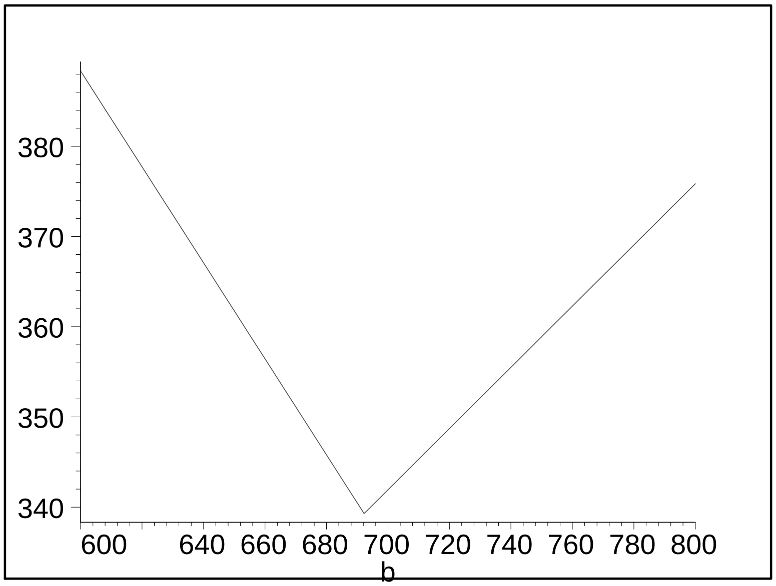
Substituting this result into condition 4, we have the problem to determine b so that

$$\sum \left| (a_i - \overline{a}) - b(\sin^2 \theta_i - \overline{\sin^2 \theta}) \right|$$

is a minimum. It is easy to see that

$$\begin{aligned} \overline{a} &= 57052.6 \\ \overline{\sin^2 \theta} &= 0.43566 \\ a &= 57052.6 - 0.43566b \end{aligned}$$

We may express $\sum |e_i|$ as $f(b) = |-301.6 + .4357b| + |-15.6 + .1369b| + |73.6 + .02914b| + |-21.4 + .1405b| + |-369.4 + .4030b|$. A graph indicates that a minimum occurs in the neighborhood of $b = 700$.



Minimizing this expression using Maple, we find that $b = 692$ and that the minimum is 339.

The corresponding corrections to be made are then

$$0, -79, 94, 76, -90$$

Finally, Boscovich computes the ellipticity as $\frac{b+2}{3a}$ for which our computations give $\frac{1}{245}$.